Solving Quadratic Equations by **Completing the Square**

REVIEW: In order to complete the square, there is only one basic prerequisite to keep in mind, that is the **square root property** which is used to solve quadratic equations of the standard form $x^2 = a$. To solve this equation, we simply take the square root of each side to obtain $x = \pm\sqrt{a}$, this is called the square root property. Below we will review two examples of solving an equation using the square root property.

Example 1: Solve $x^2 + 4 = 0$, using the square root property.

SOLUTION: First we need to put our equation in standard form, so, we subtract 4 from both sides to get $x^2 = -4$. Now that we are in standard form, we apply the square root property by taking the square root of each side, $\sqrt{x^2} = \sqrt{-4}$. We simplify this to obtain $x = \pm 2i$, therefore, $\{-2i, 2i\}$ are the two solutions of this quadratic equation.

Example 2: Solve $(x-2)^2 = 9$, using the square root property.

SOLUTION: While this problem looks a little different from the previous problem, it contains a perfect square on one side, with a number on the other, therefore it is actually already in standard form. To solve this, we apply the square root property by taking the square root on both sides, $\sqrt{(x-2)^2} = \sqrt{9}$. We simplify both sides and obtain $x - 2 = \pm 3$, now, we still need to isolate x, so we add 2 to both sides getting, $x = 2 \pm 3$. Because 2+3=5 and 2-3=-1, the solutions to this quadratic equation are $\{-1, 5\}$.

What is completing the square and why do we use it?

-Completing the square is a method for solving quadratic equations using the square root property. As you saw in the previous example, the square root property is simple to use. The problem is that to use it, your equation has to have a perfect square on one side. Completing the square is the act of forcing a perfect square on one side of the equation, and then solving it using the square root property.

-We use completing the square primarily for quadratics that cannot be factored, we will also see that completing the square leads us to the quadratic formula.

HOW TO COMPLETE THE SQUARE

-The general form of a quadratic equation is, $ax^2 + bx + c = 0$, where $a \neq 0$.

-We will divide this into two cases.

-Case 1: The equation has a leading coefficient of 1, this means $a = 1$.

-Case 2: The equation has a leading coefficient that is not 1, this means $a \neq 1$.

Case1: Completing the square for quadratics of the form $x^2 + bx + c = 0$.

1.) Isolate the two x terms by moving the c term over, $x^2 + bx = -c$.
2.) Take the b term and divide it by 2, then square it, $\left(\frac{b}{2}\right)^2$.
3.) Then add this to both sides of the equation, $x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$.
4.) The left side of your equation is now a perfect square and can be factored.
5.) Once you factor the left side, you can solve using the square root property.

TRICK: Factoring the left side of the equation is often a challenge but a handy trick to remember is that $x^2 + bx + \left(\frac{b}{2}\right)^2$ always, I repeat, always, factors as $\left(x + \frac{b}{2}\right)^2$, of course, the sign in the middle, depends on the sign of b. This may not make sense right now but see if it does as we look at a few examples.

Example 1: Solve $x^2 - 4x + 6 = 0$ by completing the square.

SOLUTION:

1.) First, we isolate the two x terms by moving 6 to the other side, $x^2 - 4x = -6$
2.) Next, we take the b term (-4), divide by 2 and square, $\left(\frac{-4}{2}\right)^2$, we can simplify this as follows, $\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$.
3.) Now, we add 4 to both sides, $x^2 - 4x + 4 = -6 + 4$, this will simplify to $x^2 - 4x + 4 = -2$.
4.) We will now factor the left side remembering the b divided by 2 is -2 for this problem. We can see that $x^2 - 4x + 4 = (x - 2)^2$. With the left side factored, our equation becomes.
$$(x - 2)^2 = -2$$
5.) We will now apply the square root property by taking the square root of each side, $\sqrt{(x-2)^2} = \sqrt{-2}$. This will simplify to, $x - 2 = \pm i\sqrt{2}$, now, all we need to do is add 2 to both sides and we will obtain $x = 2 \pm i\sqrt{2}$. The means that the solutions for this quadratic equation are $\{2 + i\sqrt{2},\ 2 - i\sqrt{2}\}$.

Example 2: Solve $x^2 + x - 6 = 0$ by completing the square.

SOLUTION:

1.) First, we isolate the two x terms by moving -6 to the other side, $x^2 + x = 6$.
2.) Next, we take the b term (1), divide by 2 and square, $\left(\frac{1}{2}\right)^2$. If we can simplify inside the parentheses, we will, but in this case, we cannot, so we will square the top and bottom separately, $\left(\frac{1}{2}\right)^2 = \frac{(1)^2}{(2)^2} = \frac{1}{4}$.
3.) Now, we add $\frac{1}{4}$, to both sides, $x^2 + x + \frac{1}{4} = 6 + \frac{1}{4}$, this will simplify to $x^2 + x + \frac{1}{4} = \frac{25}{4}$.
4.) We will now factor the left side of the equation remembering that for this problem b divided by 2 is $\left(\frac{1}{2}\right)$. We can then see that $x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$. With the left side factored, our equation becomes.
$$\left(x + \frac{1}{2}\right)^2 = \frac{25}{4}$$
5.) Now we will apply the square root property by taking the square root of each side, $\sqrt{\left(x + \frac{1}{2}\right)^2} = \sqrt{\frac{25}{4}}$. This will simplify to, $x + \frac{1}{2} = \pm\frac{5}{2}$, now, we will simply subtract $\frac{1}{2}$ from both sides to obtain $x = -\frac{1}{2} \pm \frac{5}{2}$. Since we can see that $-\frac{1}{2} + \frac{5}{2} = 2$ and $-\frac{1}{2} - \frac{5}{2} = -3$, it follows that the solutions for this quadratic equation are $\{2, -3\}$.

Case2: Completing the square for quadratics of the form $ax^2 + bx + c = 0$.

1.) Divide every term by a to make the leading coefficient 1, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.
2.) Isolate the two x terms by moving the $\frac{c}{a}$ term over, $x^2 + \frac{b}{a}x = -\frac{c}{a}$.
3.) Take the $\frac{b}{a}$ term and divide it by 2, then square it, $\left(\frac{b}{2a}\right)^2$.
4.) Then add this to both sides of the equation, $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$.
5.) The left side of your equation is now a perfect square and can be factored.
6.) Once you factor the left side, you can solve using the square root property.

TRICK: Factoring the left side of the equation is often a challenge but a handy trick to remember is that $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$ always, I repeat, always, factors as $\left(x + \frac{b}{2a}\right)^2$, of course, the sign in the middle, depends on the sign of b. This may not make sense right now but see if it does as we look at a few examples.

Example 1: Solve $2x^2 - x + 1 = 0$ by completing the square.

SOLUTION:

1.) First, we will divide out the 2 to make the leading coefficient 1.

$$x^2 - \frac{1}{2}x + \frac{1}{2} = 0$$

2.) Now we will isolate the x terms by subtracting $\frac{1}{2}$ to the other side of the equation.

$$x^2 - \frac{1}{2}x = -\frac{1}{2}$$

3.) Next, we take the b term and divide it by 2 (Remember that dividing by 2 is the same as multiplying by $\frac{1}{2}$.) and square it.

$$\left(\frac{-1/2}{2}\right)^2 = \left(-\frac{1}{2}\cdot\frac{1}{2}\right)^2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}$$

4.) We will now add $\frac{1}{16}$ to both sides of our equation.

$$x^2 - \frac{1}{2}x + \frac{1}{16} = -\frac{1}{2} + \frac{1}{16}$$

If we combine the fractions on the right side of our equation, we will get

$$x^2 - \frac{1}{2}x + \frac{1}{16} = -\frac{7}{16}.$$

5.) We remember that for this equation b divided by 2a is $-\frac{1}{4}$ so if we employ the factoring trick then the left side of our equation will factor as

$$\left(x - \frac{1}{4}\right)^2 = -\frac{7}{16}$$

6.) Now, we use the square root property to solve this equation by first taking the square root of each side.

$$\sqrt{\left(x - \frac{1}{4}\right)^2} = \sqrt{-\frac{7}{16}}$$

Simplifying each side, we will obtain,

$$x - \frac{1}{4} = \pm\frac{i\sqrt{7}}{4}$$

lastly, we add $-\frac{1}{4}$ to both sides to obtain,

$$x = \frac{1}{4} \pm \frac{i\sqrt{7}}{4}$$

This means that the solutions to our equation are $\left\{\frac{1+i\sqrt{7}}{4}, \ \frac{1-i\sqrt{7}}{4}\right\}$.

Example 2: Solve $6x^2 + x - 2 = 0$ by completing the square.

SOLUTION:

1.) First, we will divide out the 6 to make the leading coefficient 1.

$$x^2 + \frac{1}{6}x - \frac{1}{3} = 0$$

2.) Now we will isolate the x terms by adding $\frac{1}{3}$ to the other side of the equation.

$$x^2 + \frac{1}{6}x = \frac{1}{3}$$

3.) Next, we take the b term and divide it by 2 (Remember that dividing by 2 is the same as multiplying by $\frac{1}{2}$.) and square it.

$$\left(\frac{1/6}{2}\right)^2 = \left(\frac{1}{6}\cdot\frac{1}{2}\right)^2 = \left(\frac{1}{12}\right)^2 = \frac{1}{144}$$

4.) We will now add $\frac{1}{144}$ to both sides of our equation.

$$x^2 + \frac{1}{6}x + \frac{1}{144} = \frac{1}{3} + \frac{1}{144}$$

If we combine the fractions on the right side of our equation, we will get

$$x^2 + \frac{1}{6}x + \frac{1}{144} = \frac{49}{144}.$$

5.) We remember that for this equation b divided by 2a is $\frac{1}{12}$ so if we employ the factoring trick then the left side of our equation will factor as

$$\left(x + \frac{1}{12}\right)^2 = \frac{49}{144}$$

6.) Now, we use the square root property to solve this equation by first taking the square root of each side.

$$\sqrt{\left(x + \frac{1}{12}\right)^2} = \sqrt{\frac{49}{144}}$$

Simplifying each side, we will obtain,

$$x + \frac{1}{12} = \pm\frac{7}{12}$$

lastly, we subtract $\frac{1}{12}$ from both sides to obtain,

$$x = -\frac{1}{12} \pm \frac{7}{12}$$

This means that the solutions to our equation are $\left\{-\frac{2}{3}, \frac{1}{2}\right\}$.

Where does the Quadratic Formula come from?

-To answer this question, we complete the square on the quadratic $ax^2 + bx + c = 0$.

1.) First, we will divide out the a to make the leading coefficient 1.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2.) Now we will isolate the x terms by subtracting $\frac{c}{a}$ to the other side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3.) Next, we take the b term and divide it by 2 (Remember that dividing by 2 is the same as multiplying by $\frac{1}{2}$.) and square it.

$$\left(\frac{b/a}{2}\right)^2 = \left(\frac{b}{a}\cdot\frac{1}{2}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

4.) We will now add $\frac{b^2}{4a^2}$ to both sides of our equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

If we combine the fractions on the right side of our equation, we will get

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

5.) We remember that for this equation b divided by 2a is $\frac{b}{2a}$ so if we employ the factoring trick then the left side of our equation will factor as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

6.) Now, we use the square root property to solve this equation by first taking the square root of each side.

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplifying each side, we will obtain,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

lastly, we subtract $\frac{b}{2a}$ from both sides to obtain,

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

This means that if we combine these fractions because they already have a common denominator, we obtain the solution.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When you use the quadratic formula, you are really completing the square.