DeMoivre's Theorem

-You may remember working with complex numbers from your algebra days or even from precalculus

-In this tutorial, we will review the basics of complex numbers.

-We will discuss converting a complex number into polar form.

-We will cover the multiplication and division theorems.

-We will end with introducing the powerful tool of DeMoivre's Theorem.

Complex Numbers

-Complex numbers have the standard form $z = x + iy$ where x is the real part and y is the imaginary part.

-The imaginary unit i is the square root of -1, in other words $i = \sqrt{-1}$.

-This means that the imaginary unit is the only number that when squared will become negative, that is to say, $i^2 = -1$.

-Complex numbers $(x + iy)$ come in conjugate pairs $(x - iy)$.

Complex Multiplication

-With complex multiplication you simply FOIL but keep in mind that $i^2 = -1$.

-Perform the following complex multiplications

a.) $(2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2 = 2 + i - 3(-1) = 2 + i + 3 = 5 + i$

b.) $(2i)(5 + 5i) = 10i + 10i^2 = 10i + 10(-1) = 10i - 10 = -10 + 10i$

c.) $(2 - 5i)(2 + 5i) = 5 + 10i - 10i - 25i^2 = 5 - 25(-1) = 5 + 25 = 30$

Complex Division

-With complex division, you simply multiply both the numerator and denominator by the conjugate of the denominator (like rationalizing)

-Your final answer must be written with the real and imaginary parts separate.

-Perform the following complex divisions.

a.) $\frac{2-i}{3+2i} = \frac{2-i}{3+2i}\left(\frac{3-2i}{3-2i}\right) = \frac{6-4i-3i+2i^2}{9-6i+6i-4i^2} = \frac{6-7i-2}{9+4} = \frac{4-7i}{13} = \frac{4}{13} - \frac{7}{13}i$

b.) $\frac{3}{1-i} = \frac{3}{1-i}\left(\frac{1+i}{1+i}\right) = \frac{3+3i}{1+i-i-i^2} = \frac{3+3i}{1-i^2} = \frac{3+3i}{1+1} = \frac{3+3i}{2} = \frac{3}{2} + \frac{3}{2}i$

Absolute Value of Complex Numbers-The absolute value of a complex number is called the modulus.

-If $z = x + iy$ then the modulus of z is $|z| = \sqrt{x^2 + y^2}$

-Find the modulus of $z = 2 - 3i$.

$$|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

-Imagine if you had to simplify an expression like the one below.

$$\frac{\left(1 + i\sqrt{3}\right)^4\left(\sqrt{3} - i\right)^2}{(1 - i\sqrt{3})^3}$$

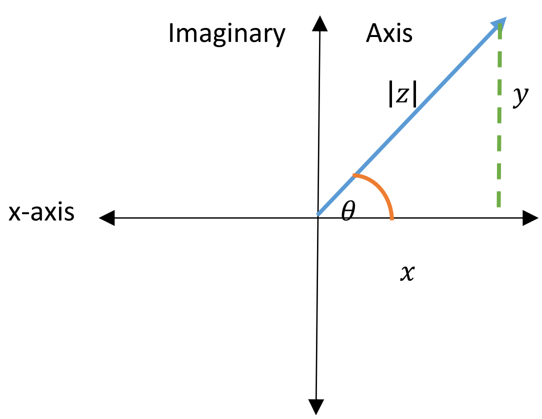
-Using the methods that we just reviewed, a problem like this would involve a lot of arithmetic.

-We are going to learn a method that will make this process much simpler using trigonometry.

Polar Form of Complex Numbers

-You have seen complex numbers in what is called rectangular form ($z = x + iy$)

-This can be plotted in rectangular coordinates by calling the y axis the imaginary axis.



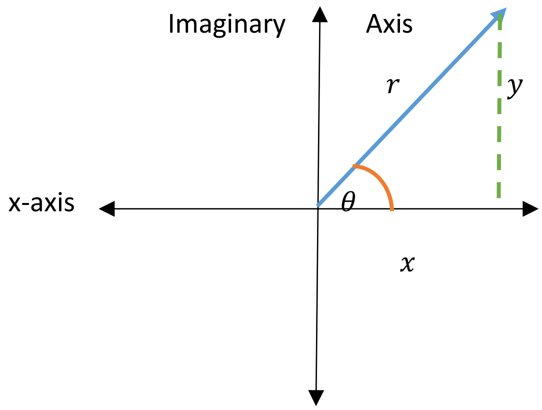
Deriving Polar Form:

-Based on the above diagram, you should notice that the modulus $|z|$ is also the radius r.

-Remember that the polar coordinate system is a system where every point is located based in its distance from the origin (r) and its angle with the positive x-axis (θ). Thus, an ordered pair is (r, θ).

-In order to put a complex number in polar form we need to take $z = x + iy$ and rewrite it in terms of r and theta.

-Let's have another at our diagram except we will replace $|z|$ with r.



-Let's start with $z = x + iy$

-Using trigonometry and the above diagram we can see that the following are true.

$$\sin\theta = \frac{y}{r} \quad \rightarrow \quad r\sin\theta = y$$

$$\cos\theta = \frac{x}{r} \quad \rightarrow \quad r\cos\theta = x$$

-This means that $x = r\cos\theta$ and $y = r\sin\theta$ and if we substitute this into $z = x + iy$, we get the following.

$$z = r\cos\theta + ir\sin\theta$$

-If we factor the r out, we get the following.

$$z = r(\cos\theta + i\sin\theta)$$

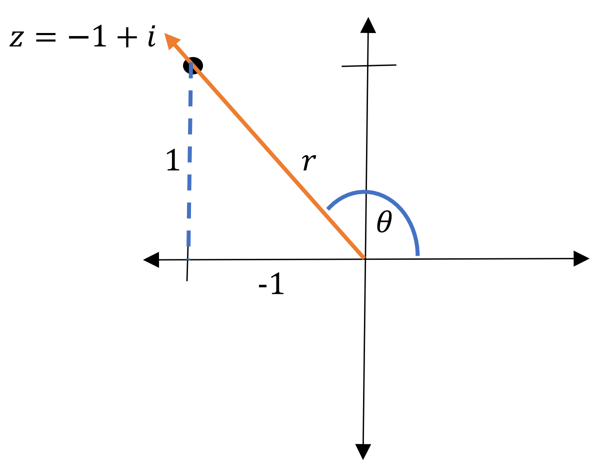
-This is the polar form for a complex number.

Example 1: Write $z = -1 + i$ in its polar form.

SOLUTION:

First, we remember that the polar form is $z = r(\cos\theta + i\sin\theta)$, this means that we simply need to find r and theta.

Let's plot the number $z = -1 + i$ on the complex plane.



-Seeing the above diagram, we know that we need to find r and θ.

-We also note very importantly that our complex number plots in the second quadrant.

-To find r, we will simply use Pythagorean theorem below.

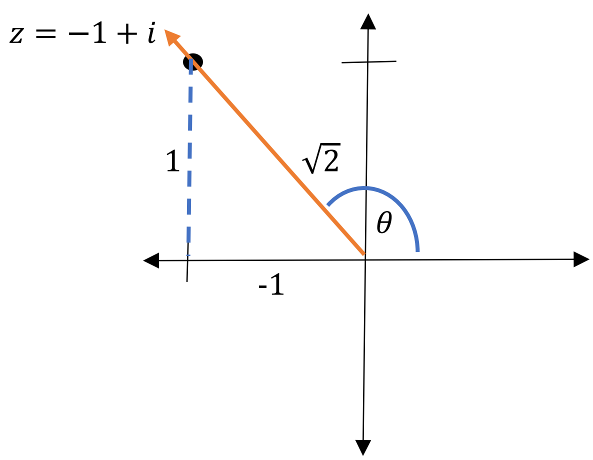
$$(-1)^2 + (1)^2 = r^2$$

$$1 + 1 = r^2$$

$$2 = r^2$$

$$\sqrt{2} = r$$

-Now that we know r, we can update our diagram.



Now that we know $r = \sqrt{2}$, we can use inverse trig to find the value of θ.

Based on the above diagram, we know that $\sin\theta = \frac{1}{\sqrt{2}}$, if we rationalize the denominator, we can say that $\sin\theta = \frac{\sqrt{2}}{2}$. Using inverse trigonometry, we can say that $\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$. If we take this to the Unit Circle, we will see that there is only one value in the second quadrant that satisfies this, and it is $\theta = 135^o$.

Now that we know that $r = \sqrt{2}$ and that $\theta = 135^o$ we can state the polar form of z below.

$$z = \sqrt{2}(\cos(135^o) + i\sin(135^o))$$

Example 2: Convert $z = 2(\cos(60^o) + i\sin(60^o))$ into standard form.

SOLUTION:

This is actually very simple; we use the unit circle to find the actual values of $\cos(60^o)$ and of $\sin(60^o)$.

$$z = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

Next distribute the 2 and you are done.

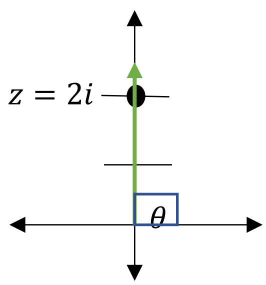
$$z = 1 + i\sqrt{3}$$

Example 3: Convert $z = 2i$ into polar form.

SOLUTION:

It is important to remember that our number is actually $z = 0 + 2i$.

Just like before, we need to plot this number in the complex plane.



This is a fun little diagram because we can get a lot of information from it.

We can clear see that z is plotted directly on the imaginary (y) axis, this means that $\theta = 90^o$.

We also remember that r is the distance from the origin, and we can clearly see that we are a distance of 2 from the origin, therefore, $r = 2$,

Therefore, the polar form of $z = 2i$ is $z = 2(\cos 90^o + i\sin 90^o)$

The Multiplication Theorem

-This theorem tells us how we can multiply two complex numbers in their polar form.

If

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

And

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then it follows that

$$z_1 \cdot z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

In English, this means that when we multiply complex numbers in polar form, we multiply the radii and add the angles.

Example 1: Calculate the following product according to the multiplication theorem.

$$\left(3(\cos 20^o + i\sin 20^o)\right)\cdot\left(4(\cos 30^o + i\sin 30^o)\right)$$

SOLUTION:

According to the multiplication theorem, we are going to multiply the radii and add the angles and simplify.

$$3\cdot 4(\cos(20^o + 30^o) + i\sin(20^o + 30^o))$$

$$12(\cos 50^o + i\sin 50^o)$$

Example 2: For the following two complex number find the product of z_1 and z_2 in standard form. Then convert both z_1 and z_2 into polar form and find their product using the multiplication theorem. Finally, convert this product from polar into standard form to show that both products are equal.

$$z_1 = 1 + i \qquad z_2 = -1 + i$$

SOLUTION:

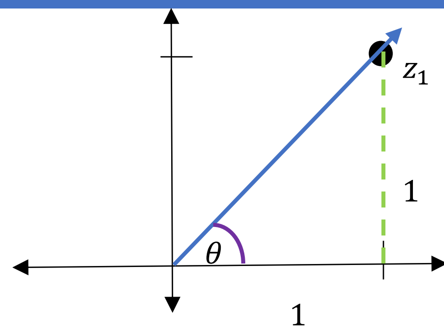
First, we find they product using FOIL.

$$z_1z_2 = (1+i)(-1+i) = -1 + i - i + i^2 = -1 + i^2 = -1 - 1 = -2$$

Next, we need to convert both numbers to Polar form so that we can find their product using the multiplication theorem.

First, we will deal with $z_1 = 1 + i$, we will start with a sketch.

Remember that x=1 and y=1 for this number.



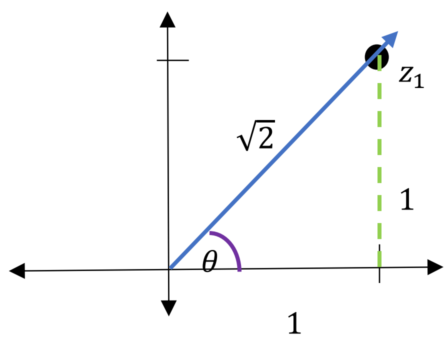
We have plotted z_1 on the complex plane and now we need to find r using the Pythagorean theorem.

$$1^2 + 1^2 = r^2$$

$$2 = r^2$$

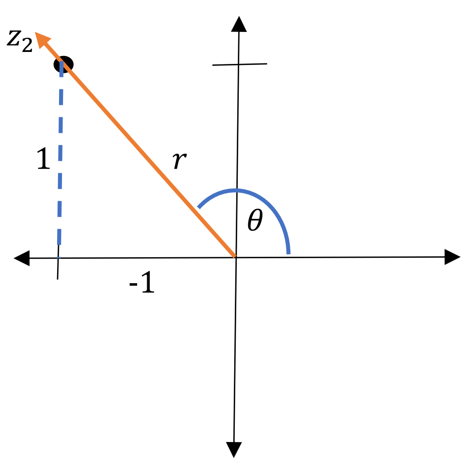
$$\sqrt{2} = r$$

Now we can complete the above diagram below.



Since we know r, we can determine that $\sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and this means that $\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ and if we take this to the unit circle, there is only one value in the 1st quadrant that will work and it is $\theta = 45^o$. Since we know r and θ, we can say the $z_1 = \sqrt{2}(\cos 45^o + i \sin 45^o)$.

Next, we will convert $z_2 = -1 + i$ into Polar form, first, we plot z_2 on the complex plane.



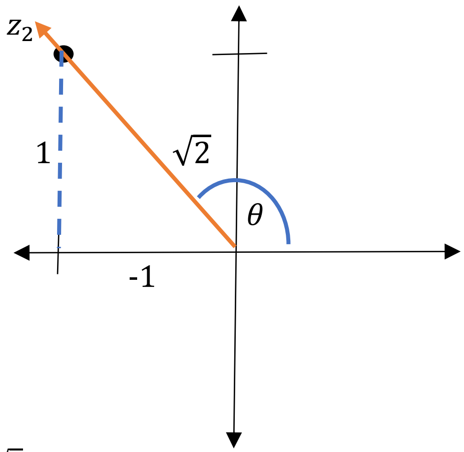
Next, we use Pythagorean theorem to find the value of r.

$$(-1)^2 + (1)^2 = r^2$$

$$2 = r^2$$

$$\sqrt{2} = r$$

So, now we can complete the diagram below.



We can see that $\sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and with inverse trigonometry we get $\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$. According to the unit circle in quadrant 2 the solution is $\theta = 135^o$.

This means that $z_2 = \sqrt{2}(\cos 135^o + i\sin 135^o)$.

We now know that $z_1 = \sqrt{2}(\cos 45^o + i\sin 45^o)$ and $z_2 = \sqrt{2}(\cos 135^o + i\sin 135^o)$.

According to the multiplication theorem

$$z_1 z_2 = \sqrt{2}\cdot\sqrt{2}(\cos(45^o + 135^o) + i\sin(45^o + 135^o))$$

$$z_1 z_2 = 2(\cos 180^o + i\sin 180^o)$$

We can see that the product is $z_1 z_2 = 2(\cos 180^o + i\sin 180^o)$ in Polar form, lastly, we are asked to convert it into standard form to see if we get the same answer that we did when we FOILed.

$$z_1 z_2 = 2(\cos 180^o + i\sin 180^o)$$

$$z_1 z_2 = 2(-1 + i\cdot 0)$$

$$z_1 z_2 = 2(-1 + 0)$$

$$z_1 z_2 = -2$$

We got the same answer!

The Division Theorem

-This theorem will tell us how to divide complex numbers in their polar form.

If

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

And

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then their quotient z_1/z_2 is as follows.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

In English this means that when we divide complex numbers in polar form, we divide the radii and subtract the angles.

Example 1: Divide the following complex numbers using the division theorem.

$$\frac{20(\cos 75^o + i\sin 75^o)}{5(\cos 40^o + i\sin 40^o)}$$

SOLUTION:

We know from the division theorem that we need to divide the radii and subtract the angles and then simplify as shown below.

$$\frac{20}{5}[\cos(75^o - 40^o) + i\sin(75^o - 40^o)]$$

$$4[\cos(35^o) + i\sin(35^o)]$$

Example 2: For the following two complex numbers in standard form find the quotient z_1/z_2 by rationalizing. After this, convert both complex numbers into polar form and divide them using the division theorem. Finally convert your answer back into standard form to show that you have gotten the same solution.

$$z_1 = \sqrt{3} + i \qquad z_2 = 2i$$

SOLUTION:

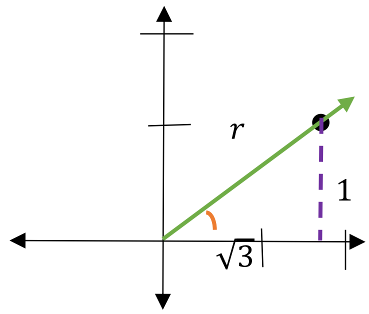
The first thing that we will do is find the quotient by rationalizing, so we multiply both top and bottom by the conjugate of the denominator.

$$\frac{\sqrt{3}+i}{2i} = \frac{\sqrt{3}+i}{2i}\cdot\left(\frac{-2i}{-2i}\right) = \frac{-2i\sqrt{3} - 2i^2}{-4i^2} = \frac{-2i\sqrt{3} - 2(-1)}{-4(-1)} = \frac{-2i\sqrt{3} + 2}{4} = \frac{2 - 2i\sqrt{3}}{4}$$

$$= \frac{2}{4} - \frac{2i\sqrt{3}}{4} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Our next task is to convert these two complex numbers into polar form.

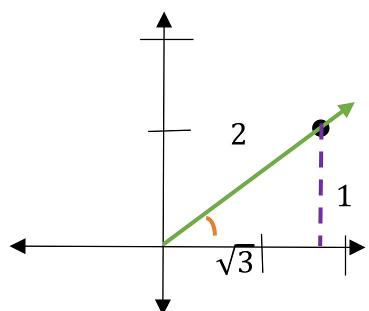
We start with $z_1 = \sqrt{3} + i$. Of course, the first thing that we will do is sketch a diagram.



In order to find r, we will use Pythagorean theorem below.

$$\left(\sqrt{3}\right)^2 + (1)^2 = (r)^2$$
$$3 + 1 = r^2$$
$$4 = r^2$$
$$2 = r$$

Since r=2, we can update our diagram below.

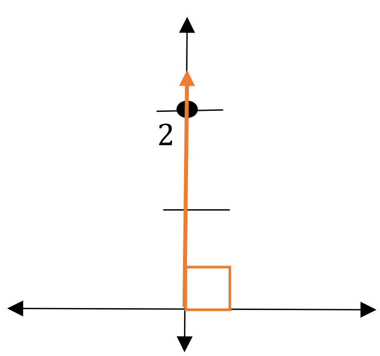


Since we know r, we can say that $\sin\theta = \frac{1}{2}$, this means that $\theta = \sin^{-1}\left(\frac{1}{2}\right)$.

If we take this to the first quadrant of the unit circle, we will see that $\theta = 30^o$.

This means that $z_1 = 2(\cos 30^o + i \sin 30^o)$.

Next, we turn our attention to $z_2 = 2i$, we start with a diagram.



We can see that since this number is actually on the imaginary axis, θ must be 90^o and we can also see that r (the distance from the origin) is 2.

Therefore, it becomes clear that $z_2 = 2(\cos 90^o + i \sin 90^o)$.

Since we now know that $z_1 = 2(\cos 30^o + i \sin 30^o)$ and $z_2 = 2(\cos 90^o + i \sin 90^o)$, we can apply the division theorem.

$$\frac{z_1}{z_2} = \frac{2(\cos 30^o + i \sin 30^o)}{2(\cos 90^o + i \sin 90^o)} = \frac{2}{2}[\cos(30^o - 90^o) + i \sin(30^o - 90^o)]$$

If we simplify this, we get $1[\cos(-60^o) + i \sin(-60^o)]$.

Does the negative angle look strange? It very well may but it is nothing to worry about, we can simply add 360 to both angles to get their coterminal versions.

$$[\cos(-60^o) + i \sin(-60^o)] \quad \rightarrow \quad [\cos 300^o + i \sin 300^o]$$

Now that our polar complex number looks a little more respectable, we can take each value to the unit circle to convert our number to standard form.

$$\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)$$

If we distribute, we get the following final solution.

$$\frac{z_1}{z_2} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

This is the same as the previous answer we got!

DeMoivre's Theorem

-At last, we come to the grand finale!

-DeMoivre's Theorem, while relatively simple, has many far-reaching consequences that often arise in advanced mathematics.

-What the Precalculus students' needs this for is to simplify expressions where a complex number is being raised to a power.

-If I asked you to calculate $\left(-1 - i\sqrt{3}\right)^{12}$, you could do it by writing the expression out 12 times and doing quite a bit of FOIL, however DeMoivre's Theorem shows us an easier way to simplify the expression.

-DeMoivre's Theorem states that if $z = r(\cos\theta + i\sin\theta)$ is a complex number in polar form and n is an integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[r(\cos n\theta + i\sin n\theta)]$$

In English this means that when we raise a complex number to the power of n, we can raise the radius to that power and multiply the angles by that number.

Example 1: Find $[2(\cos 10^o + i\sin 10^o)]^6$ using DeMoivre's Theorem.

SOLUTION:

If we apply the theorem, we get the following.

$$[2(\cos 10^o + i\sin 10^o)]^6 = 2^6[(\cos 6\cdot 10^o + i\sin 6\cdot 10^o)]$$

If we simplify this, we get the following solution.

$$64[(\cos 60^o + i\sin 60^o)]$$

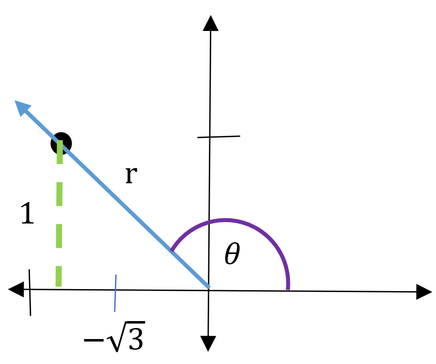
And were done!

Example 2: Use DeMoivre's Theorem to find $\left(-\sqrt{3} + i\right)^4$

SOLUTION:

In order to use DeMoivre's theorem, our number needs to be in polar form, so we are going to convert our number to polar form, and then apply the theorem and simplify, after this is complete, we will return our final answer to standard form.

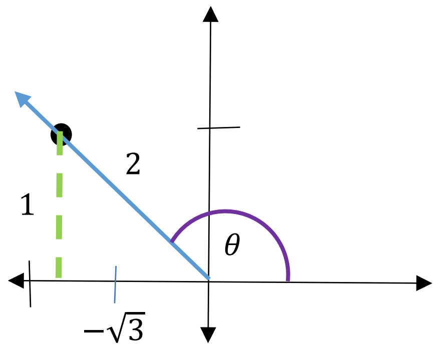
1- We convert $-\sqrt{3} + i$ to polar form.



-Now, we simply need to use Pythagorean theorem to find the value of r.

$$(1)^2 + (-\sqrt{3})^2 = r^2$$
$$1 + 3 = r^2$$
$$4 = r^2$$
$$2 = r$$

Now, we can update our diagram.



Now that we know r, we can see that $\sin\theta = \frac{1}{2}$ and $\theta = \sin^{-1}\left(\frac{1}{2}\right)$, if we go to the 2nd quadrant of the unit circle, we can see that $\theta = 150^o$.

This means that $-\sqrt{3} + i = 2(\cos 150^o + i\sin 150^o)$

Now, we remember that we are raising this to the 4th power, therefore.

$$[2(\cos 150^o + i\sin 150^o)]^4$$

2- Now we can apply DeMoivre's Theorem and simplify
$$[2(\cos 150^o + i\sin 150^o)]^4 = 2^4[\cos(4\cdot 150^o) + i\sin(4\cdot 150^o)]$$
This will become.
$$16[\cos 600^o + i\sin 600^o]$$
However, 600 is not on the unit circle, but we can subtract 360 from it to put the angle within the range of 0 to 360.
$$16[\cos 240^o + i\sin 240^o]$$

3- Lastly, we will convert $16[\cos 240^o + i\sin 240^o]$ into standard form.

$$16\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$16\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$-8 - 8i\sqrt{3}$$

This is the final answer! While it might seem like a lot of work, the elegance of this method is impossible to deny.

Final Example: Convert all complex numbers into standard form and them simplify the expression, lastly convert your answer back into standard form.

$$\frac{\left(1+i\sqrt{3}\right)^4\left(\sqrt{3}-i\right)^2}{(1-i\sqrt{3})^3}$$

Note: In this example, we will not cover converting the numbers to polar form, that exercise is performed in appendix.

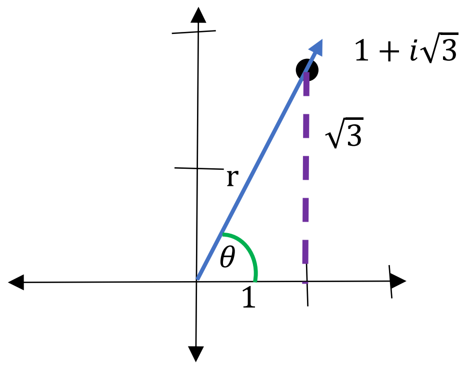
SOLUTION:

	$\frac{\left(1+i\sqrt{3}\right)^4\left(\sqrt{3}-i\right)^2}{(1-i\sqrt{3})^3}$
Convert every number to Polar	$\frac{\left(2(\cos 60^o + i\sin 60^o)\right)^4\left(2(\cos 330^o + i\sin 330^o)\right)^2}{\left(2(\cos 300^o + i\sin 300^o)\right)^3}$
Apply DeMoivre's Theorem	$\frac{2^4(\cos 4\cdot 60^o + i\sin 4\cdot 60^o)\cdot 2^2(\cos 2\cdot 330^o + i\sin 2\cdot 330^o)}{2^3(\cos 3\cdot 300^o + i\sin 3\cdot 300^o)}$
Simplify	$\frac{16(\cos 240^o + i\sin 240^o)\cdot 4(\cos 660^o + i\sin 660^o)}{8(\cos 900^o + i\sin 900^o)}$
Simplify the numerator using the multiplication theorem.	$\frac{16\cdot 4(\cos(240^o + 660^o) + i\sin(240^o + 660^o))}{8(\cos 900^o + i\sin 900^o)}$
Simplify	$\frac{64(\cos 900^o + i\sin 900^o)}{8(\cos 900^o + i\sin 900^o)}$
Apply the Division Theorem	$\frac{64}{8}[\cos(900^o - 900^o) + i\sin(900^o - 900^o)]$
Simplify	$8(\cos 0^o + i\sin 0^o)$
Convert to Standard Form	$8(1 + i(0))$
Simplify	8

You must admit, its pretty cool that we can simplify a complicated expression in only a few simple steps.

Appendix

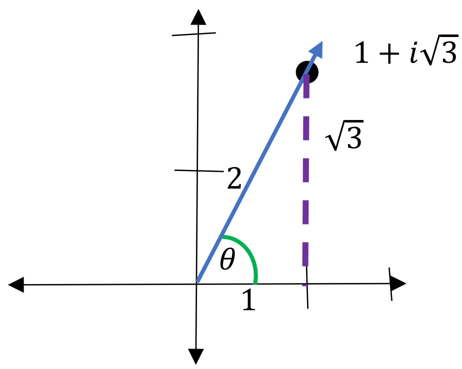
a.) Convert $1 + i\sqrt{3}$ into polar form.
We will start by plotting this number in the complex plane.



We can find the r using Pythagorean theorem.

$$(1)^2 + \left(\sqrt{3}\right)^2 = r^2$$
$$1 + 3 = r^2$$
$$4 = r^2$$
$$2 = r$$

Now we can rewrite our diagram below.



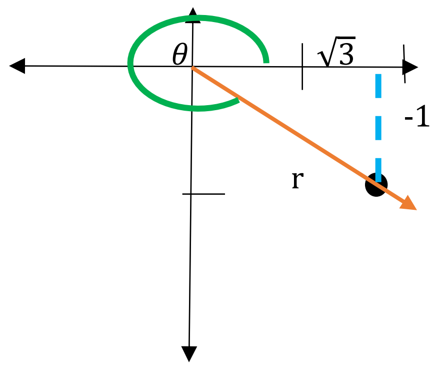
We can see that our angle is in the first quadrant and we can also see that $\sin\theta = \frac{\sqrt{3}}{2}$.

If we go to the first quadrant of the unit circle, we see that $\theta = 60^o$.

This means that $1 + i\sqrt{3} = 2(\cos 60^o + i \sin 60^o)$.

b.) Convert $\sqrt{3} - i$ into polar form.

We need to remember that $x = \sqrt{3}$ and $y = -1$ and we will draw a sketch below.



We will use Pythagorean theorem to find the value of r.

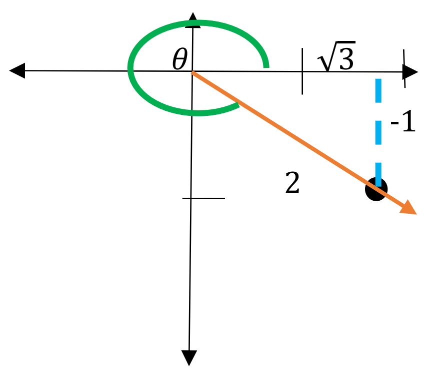
$$\left(\sqrt{3}\right)^2 + (-1)^2 = r^2$$

$$3 + 1 = r^2$$

$$4 = r^2$$

$$2 = r$$

We can now update our diagram below.

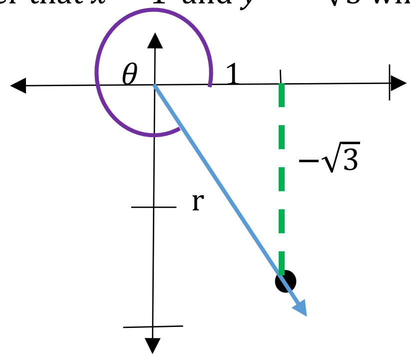


We can now see that $\sin\theta = -\frac{1}{2}$ and if we take that to the 3rd quadrant of the unit circle, we find that $\theta = 330^o$.

This means that $\sqrt{3} - i = 2(\cos 330^o + i \sin 330^o)$.

c.) Convert $1 - i\sqrt{3}$ into Polar Form.

In this case we remember that $x = 1$ and $y = -\sqrt{3}$ when we draw our sketch below.



We can use Pythagorean theorem below to find the value of r.

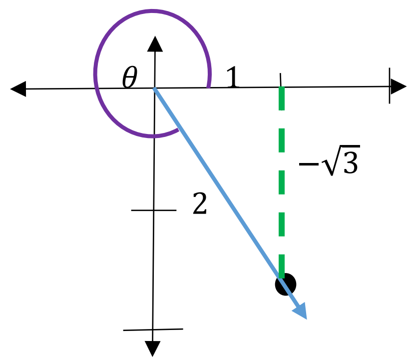
$$(1)^2 + \left(-\sqrt{3}\right)^2 = r^2$$

$$1 + 3 = r^2$$

$$4 = r^2$$

$$2 = r$$

Since we know the value of r, we can update our diagram below.



We can see that $\sin\theta = -\frac{\sqrt{3}}{2}$ and if we take this to the 4th quadrant of the unit circle, we will see that $\theta = 300^o$.

This means that $1 - i\sqrt{3} = 2(\cos 300^o + i\sin 300^o)$