Gauss-Jordan Elimination

-Gauss-Jordan elimination is a method very similar to Gaussian elimination.

-We still put our system of equations into an augmented and use row transformations.

-The difference in that instead of using the row transformations to create a lower triangular matrix, we use the row transformations to create an identity matrix in the variable columns of the matrix.

-When we do this, we get all our solutions in the solution column

Steps for Gauss-Jordan Elimination (in 3 variables)

1.) Put your system in an augmented matrix.
2.) Convert the first number in the first column into a 1.
3.) Pivot off that 1 to turn the remaining numbers in that column into 0s.
4.) Convert the middle number in the second column into a 1.
5.) Pivot off that 1 to turn remaining numbers in the column into 0s.
6.) Convert the last number in the 3rd column into a 1.
7.) Pivot off that 1 to turn the remaining numbers in that column into 0s.
8.) Take your solutions from the last column.

Example 1: Solve the following system of equations using Gauss-Jordan elimination.

$$\begin{cases} x + y + z = 0 \\ x + 2y - 3z = 5 \\ 3x + 4y + 2z = -1 \end{cases}$$

SOLUTION:

First, we will put this system into an augmented matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -3 & 5 \\ 3 & 4 & 2 & -1 \end{bmatrix}$$

Second, the first number in the first column in already one.

Third, we will pivot off that 1 to make the two numbers below in the first column into 0s.

$$-R_1 + R_2 \to R_2 \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 5 \\ 3 & 4 & 2 & -1 \end{bmatrix}$$

$$-3R_1 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 5 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

Fourth, since the middle number in the second column is already 1, we can move on.

Fifth, we need to pivot off that 1 to make the numbers above and below into 0s.

$$-R_2 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & -4 & 5 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$-R_2 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

Sixth, we need to turn the last number in column 3 into a 1.

$$\frac{1}{3}R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Seventh, we need to pivot off that 1 and convert the two numbers above it into 0s.

$$4R_3 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$-5R_3 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Eighth, this means that the solution to our system is $(5, -3, -2)$.

Example 2: Use Gauss-Jordan elimination to solve the following system of equations.

$$\begin{cases} 2x + y - 2z = -1 \\ 3x - 3y - z = \ \ 5 \\ x - 2y + 3z = \ \ 6 \end{cases}$$

SOLUTION:

First, we are going to put our system into an augmented matrix.

$$\begin{bmatrix} 2 & 1 & -2 & -1 \\ 3 & -3 & -1 & 5 \\ 1 & -2 & 3 & 6 \end{bmatrix}$$

Second, the first number in the first column needs to be a 1, since we have a 1 in the 3rd row, we will flip those two rows.

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & -2 & 3 & 6 \\ 3 & -3 & -1 & 5 \\ 2 & 1 & -2 & -1 \end{bmatrix}$$

Third, we will pivot off that 1 to make the two numbers below it into 0s.

$$-3R_1 + R_2 \to R_2 \quad \begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 3 & -10 & -13 \\ 2 & 1 & -2 & -1 \end{bmatrix}$$

$$-2R_1 + R_3 \to R_3 \quad \begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 3 & -10 & -13 \\ 0 & 5 & -8 & -13 \end{bmatrix}$$

Fourth, we will turn the middle number of the second column into a 1.

$$\frac{1}{3}R_2 \to R_2 \quad \begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 1 & -{}^{10}/_{3} & -{}^{13}/_{3} \\ 0 & 5 & -8 & -13 \end{bmatrix}$$

Fifth, we will pivot off that 1 to make the numbers above and below it into 0s.

$$2R_2 + R_1 \to R_1 \quad \begin{bmatrix} 1 & 0 & -{}^{11}/_{3} & -{}^{8}/_{3} \\ 0 & 1 & -{}^{10}/_{3} & -{}^{13}/_{3} \\ 0 & 5 & -8 & -13 \end{bmatrix}$$

$$-5R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 0 & -{}^{11}/_{3} & -{}^{8}/_{3} \\ 0 & 1 & -{}^{10}/_{3} & -{}^{13}/_{3} \\ 0 & 0 & {}^{26}/_{3} & {}^{26}/_{3} \end{bmatrix}$$

Sixth, we will turn the final number in the third column into a 1.

$$\frac{3}{26}R_3 \to R_3 \quad \begin{bmatrix} 1 & 0 & -{}^{11}/_{3} & -{}^{8}/_{3} \\ 0 & 1 & -{}^{10}/_{3} & -{}^{13}/_{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Seventh, we will pivot off that 1 to make the two numbers above it into 0s.

$$\frac{10}{3}R_3 + R_2 \to R_2 \quad \begin{bmatrix} 1 & 0 & -{}^{11}/_{3} & -{}^{8}/_{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\frac{11}{3}R_3 + R_1 \to R_1 \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Eighth, we can see that the solution to the given system is $(1, -1, 1)$.

HINT: Never assume that getting odd looking fractions means that you have made a mistake.

Example 3: Use Gauss-Jordan elimination to solve the following system of equations.

$$\begin{cases} 2x + 3y + 7z = \ \ 13 \\ 3x + 2y - 5z = -22 \\ 5x + 7y - 3z = -28 \end{cases}$$

SOLUTION:

First, we put our system into an augmented matrix.

$$\begin{bmatrix} 2 & 3 & 7 & 13 \\ 3 & 2 & -5 & -22 \\ 5 & 7 & -3 & -28 \end{bmatrix}$$

Second, we make the first number in the first column 1 by multiplying by ½ .

$$\frac{1}{2}R_1 \to R_1 \qquad \begin{bmatrix} 1 & 3/2 & 7/2 & 13/2 \\ 3 & 2 & -5 & -22 \\ 5 & 7 & -3 & -28 \end{bmatrix}$$

Third, we pivot off that 1 to make the two number below into 0s.

$$-3R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & 3/2 & 7/2 & 13/2 \\ 0 & -5/2 & -31/2 & -83/2 \\ 5 & 7 & -3 & -28 \end{bmatrix}$$

$$-5R_1 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & 3/2 & 7/2 & 13/2 \\ 0 & -5/2 & -31/2 & -83/2 \\ 0 & -1/2 & -41/2 & -121/2 \end{bmatrix}$$

Fourth, we turn the middle number in the second column into a 1.

$$-\frac{2}{5}R_2 \to R_2 \qquad \begin{bmatrix} 1 & 3/2 & 7/2 & 13/2 \\ 0 & 1 & 31/5 & 83/5 \\ 0 & -1/2 & -41/2 & -121/2 \end{bmatrix}$$

Fifth, we pivot of that 1 to turn the numbers above and below it into 0s.

$$-\frac{3}{2}R_2 + R_1 \to R_1 \qquad \begin{bmatrix} 1 & 0 & -29/5 & -92/5 \\ 0 & 1 & 31/5 & 83/5 \\ 0 & -1/2 & -41/2 & -121/2 \end{bmatrix}$$

$$\frac{1}{2}R_2 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 0 & -{}^{29}/_5 & -{}^{92}/_5 \\ 0 & 1 & {}^{31}/_5 & {}^{83}/_5 \\ 0 & 0 & -{}^{87}/_5 & -{}^{261}/_5 \end{bmatrix}$$

Sixth, we turn the last number in the 3rd column into a 1.

$$-\frac{5}{87}R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 0 & -{}^{29}/_5 & -{}^{92}/_5 \\ 0 & 1 & {}^{31}/_5 & {}^{83}/_5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Seventh, we pivot off that 1 to convert the two number above it into 0s.

$$-\frac{31}{5}R_3 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 0 & -{}^{29}/_5 & -{}^{92}/_5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\frac{29}{5}R_3 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Eighth, we can see that the solution to our system is $(-1, -2, 3)$.

LAST NOTE:

For these problems, if you have 10 steps, you can have 10 mistakes. It is also true that one mistake can often snowball and throw off your entire solution. Please always double check your work and try to be as neat as possible with organized steps.