Gaussian Elimination

-One method that is often used to make solving systems of equations in a simple way is called Gaussian Elimination.

-You take your system of equations and put it in an augmented matrix.

-Then, you use row transformations to reduce the matrix until it becomes a lower triangular matrix.

Augmented Matrix

-When you put a system of equations into a matrix, it is called an augmented matrix.

-Below is an example of what this looks like.

$$\begin{cases} 2x - 3y = 9 \\ 5x + 7y = 3 \end{cases} \quad \rightarrow \quad \begin{bmatrix} 2 & -3 & 9 \\ 5 & 7 & 3 \end{bmatrix}$$

-See how this works? This first column is for x, the second is for y, and the third is for the solution.

Lower Triangular Matrix

-A matrix that a lower triangle of 0s.

-While it is not necessary, we will insist in this tutorial that the main diagonal should be 1s.

-Below are three examples of lower triangular matrices.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 4 & -2 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

-Our goal is to convert an augmented matrix into a lower triangular matrix using row transformations.

Row Transformations

-It can be difficult to explain row transformations and their notation, so we are going to look at examples that will hopefully clarify the concept.

$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & -1 & 2 \end{bmatrix} \quad R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & -3 & 5 \\ 4 & -4 & 7 \end{bmatrix}$$

-In this example, we add row 1 to row 2 effective on row 2. Note that only row 2 has been changed.

$$\begin{bmatrix} 2 & 3 & -4 \\ 1 & -2 & 4 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & -4 \end{bmatrix}$$

-In this case we flipped rows 1 and 2.

$$\begin{bmatrix} -2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad -\frac{1}{2}R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & -{}^{1}/_{2} & -{}^{3}/_{2} \\ 2 & 3 & 1 \end{bmatrix}$$

-In this case, we multiplied row 1 by -1/2 effective on row 1, in order to turn the first number into a 1.

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \end{bmatrix} \quad -5R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -14 \end{bmatrix}$$

-In this case, we multiplied the first row by -5 and added it to the second row.

-These are the basic types of row transformations that we are going to use for Gaussian Elimination.

Gaussian Elimination Steps

1.) Put your system of equations into an augmented matrix
2.) Convert the top number in the 1st column into 1.
3.) Use row transformations to convert the two numbers beneath 1 in the first column into 0s.
4.) Convert the middle number in the second column to a 1.
5.) Use this to turn the number below it into a 0.
6.) Convert the bottom number in the 3rd column into a 1.
7.) Take your system of equations out of the matrix.
8.) Use back substitution to find all the solutions to the system.

Example 1: Solve the following system using Gaussian elimination.

$$\begin{cases} 2x + y + 2z = 2 \\ 3x - 5y - z = 4 \\ x - 2y - 3z = -6 \end{cases}$$

SOLUTION: First, we put the system into an augmented matrix.

$$\begin{bmatrix} 2 & 1 & 2 & 2 \\ 3 & -5 & -1 & 4 \\ 1 & -2 & -3 & -6 \end{bmatrix}$$

Second, we need to make the top number in the first column 1, we can do this by flipping rows 1 and 3.

$$R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & -2 & -3 & -6 \\ 3 & -5 & -1 & 4 \\ 2 & 1 & 2 & 2 \end{bmatrix}$$

Third, we are going to pivot off of the 1 to make the two numbers below it into 0s.

$$-3R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & -2 & -3 & -6 \\ 0 & 1 & 8 & 22 \\ 2 & 1 & 2 & 2 \end{bmatrix}$$

$$-2R_1 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & -2 & -3 & -6 \\ 0 & 1 & 8 & 22 \\ 0 & 5 & 8 & 14 \end{bmatrix}$$

Fourth, we need to make the middle number in the second column a 1, since this is already true, we move to the 5^{th} step.

Fifth, we need to pivot off that 1 to convert the 5 beneath it into a 0.

$$-5R_2 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & -2 & -3 & -6 \\ 0 & 1 & 8 & 22 \\ 0 & 0 & -32 & -96 \end{bmatrix}$$

Sixth, we need to convert the last number in the 3^{rd} column into a 1.

$$-\frac{1}{32}R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & -2 & -3 & -6 \\ 0 & 1 & 8 & 22 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Seventh, now that we have a lower triangular matrix, we can take our system out of the matrix.

$$\begin{cases} x - 2y - 3z = -6 \\ \qquad y + 8z = 22 \\ \qquad\qquad z = 3 \end{cases}$$

Eighth, we already know z, to find y, we substitute into the second equation.

$$y + 8(3) = 22 \quad \rightarrow y + 24 = 22 \quad \rightarrow \quad y = -2$$

Next, we substitute z and y into the first equation.

$$x - 2(-2) - 3(3) = -6 \quad \rightarrow \quad x + 4 - 9 = -6 \quad \rightarrow \quad x - 5 = -6 \quad \rightarrow \quad x = -1$$

Now, we have all of the solutions to our system of equation and we can write this as the ordered triple $(-1, -2, 3)$.

Example 2: Use Gaussian Elimination to solve the following system of equations.

$$\begin{cases} 2x + 2y \ + 7z = -1 \\ 2x + \ y + 2z = 2 \\ 4x + 6y + \ \ z = 15 \end{cases}$$

SOLUTION:

First, we put this system into an augmented matrix.

$$\begin{bmatrix} 2 & 2 & 7 & -1 \\ 2 & 1 & 2 & 2 \\ 4 & 6 & 1 & 15 \end{bmatrix}$$

Second, we convert the first number in the first column into a 1.

$$\frac{1}{2}R_1 \to R_1 \quad \begin{bmatrix} 1 & 1 & {}^7/_2 & -{}^1/_2 \\ 2 & 1 & 2 & 2 \\ 4 & 6 & 1 & 15 \end{bmatrix}$$

Third, we will pivot off that 1 in the first column to make the 2 and 4 beneath it into 0s.

$$-2R_1 + R_2 \to R_2 \quad \begin{bmatrix} 1 & 1 & {}^7/_2 & -{}^1/_2 \\ 0 & -1 & -5 & 3 \\ 4 & 6 & 1 & 15 \end{bmatrix}$$

$$-4R_1 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & {}^7/_2 & -{}^1/_2 \\ 0 & -1 & -5 & 3 \\ 0 & 2 & -13 & 17 \end{bmatrix}$$

Fourth, we will convert the middle number in the second column to a 1.

$$-R_2 \to R_2 \quad \begin{bmatrix} 1 & 1 & {}^7/_2 & -{}^1/_2 \\ 0 & 1 & 5 & -3 \\ 0 & 2 & -13 & 17 \end{bmatrix}$$

Fifth, we pivot off the middle 1 in column 2 and convert the 2 beneath it into a 0.

$$-2R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & {}^7/_2 & -{}^1/_2 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -23 & 23 \end{bmatrix}$$

Sixth, we convert the last number in the third column into a 1.

$$-\frac{1}{23}R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & {}^7/_2 & -{}^1/_2 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Seventh, we take our system of equations out of the augmented matrix.

$$\begin{cases} x + y + \frac{7}{2}z = -\frac{1}{2} \\ y + 5z = -3 \\ z = -1 \end{cases}$$

Eighth, we use back substitution to get our other two solutions.

We plug z into the second equation to get y.

$$y + 5(-1) = -3 \quad \rightarrow \quad y - 5 = -3 \quad \rightarrow \quad y = 2$$

Lastly, we plug z and y into the first equation and solve for x.

$$x + (2) + \frac{7}{2}(-1) = -\frac{1}{2} \quad \rightarrow \quad x + 2 - \frac{7}{2} = -\frac{1}{2} \quad \rightarrow \quad x - \frac{3}{2} = -\frac{1}{2} \quad \rightarrow \quad x = 1$$

Now that we know all the solutions we can place them in the ordered triple $(1, 2, -1)$.