

Introduction to Logarithms

-A logarithm is the inverse function for an exponent; therefore, we will review exponential functions first.

Review of Exponential Functions

-An exponential function has the general form $f(x) = b^x$, where $0 < b < 1$, or $b > 1$.

-b is called the **base** and x is called the **exponent**.

-Let's see how an exponential function works, we will look at the function $f(x) = 2^x$, in the table below.

x	-2	-1	0	1	2
$f(x)$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 1$	$2^2 = 4$

-An interesting thing that we can notice is that even when **we input a negative number into the function, it never outputs a negative**. This will always be true for exponents unless some transformation is imposed.

-We should also note that **whenever we input 0 into the function, it outputs 1**. This will always be true for exponents unless some transformation is imposed.

A Logarithm is the inverse function for an Exponent

-We remember that inverse functions do the exact opposite of one another.

-An example can be seen in the table above; the exponential function sends -2 to $\frac{1}{4}$. The logarithm would send $\frac{1}{4}$ back to -2 .

-Inverse functions undo one another and this concept is going to be crucial to calculating logarithms.

-An exponential function $f(x) = b^x$ with a base of b, will have an inverse function which is a logarithm that also has a base of b, we will write it as $f^{-1}(x) = \log_b(x)$.

-In the table below, see some examples of exponential functions and their inverse logarithms.

Function	Inverse Function
4^x	$\log_4(x)$
$\left(\frac{1}{2}\right)^x$	$\log_{\frac{1}{2}}(x)$
7^x	$\log_7(x)$

Logarithmic Functions

-A previously stated a logarithmic function with a base of b is $f(x) = \log_b(x)$, where $0 < b < 1$, or $b > 1$.

-Because an exponent will never have negative output (see above), a logarithm can **never** have negative input.

-Because when you raise a number to the 0 power, you get 1, whenever, I repeat, whenever, you put 1 inside a logarithm, you get 0. In other words, $\log_b(1) = 0$.

-Because the most common logarithm is log base 10, for log base 10 we simply write $\log(x)$. If you see a logarithm that doesn't have a base, that means that it is base 10.

-Another common logarithm is $\log_e(x)$, because this occurs so often in natural growth patterns, it is called the natural logarithm and abbreviated $\ln(x)$.

-Logarithms, much like exponents are often used to model types of growth.

Using Logarithms in Equations

-While logarithmic functions are extremely valuable in many areas of applied mathematics and science, they are also a very powerful problem-solving tool.

-We are going to cover how to use logarithms in equations.

Convert the following exponential equations to logarithmic equations

a.) Convert $2^3 = 8$ into a logarithmic equation.

Solution:

First, since the exponent has a base of 2, we will use a logarithm with a base of 2 on both sides to undo the exponent.

$$\log_2(2^3) = \log_2(8)$$

On the left side, the log base 2 and the 2 cancel because they are inverse functions, the only term that remains is the 3, giving us.

$$3 = \log_2(8)$$

b.) Convert $10^2 = 100$ into a logarithmic equation.

Solution:

First, since the exponent has a base of 10, we will use a log base 10 on both sides to undo the exponent.

$$\log(10^2) = \log(100)$$

On the left side, the log base 10 and the 10 cancel because they are inverse functions, the only term that remains is the 2 giving us $2 = \log(100)$.

c.) Convert $5^{-3} = \frac{1}{125}$ into a logarithmic equation.

Solution:

First, since the exponent has a base of 5, we will use a log base 5 on both sides to undo the exponent.

$$\log_5(5^{-3}) = \log_5\left(\frac{1}{125}\right)$$

On the left side, the log base 5 and the 5 cancel because they are inverse functions, the only term that remains is -3, this gives us a logarithmic equation of.

$$-3 = \log_5\left(\frac{1}{125}\right)$$

d.) Convert $\sqrt{4} = 2$ into a logarithmic equation.

Solution:

While this may look a little different to start with, remember that a square root is an exponent and we can rewrite the equation like this, $4^{\frac{1}{2}} = 2$. Now, since our exponent has a base of 4, we will use a log base 4 on both sides to undo the exponent.

$$\log_4\left(4^{\frac{1}{2}}\right) = \log_4(2)$$

On the left side, the log base 4 and the 4 cancels because they are inverse functions, the only term that remains on that side is the $\frac{1}{2}$, this gives the following logarithmic equation.

$$\frac{1}{2} = \log_4(2)$$

e.) Convert $e^x = 5$ into a logarithmic equation.

Solution:

We need to use a log base e (natural logarithm) for this problem, so we will apply ln on both sides to undo the e.

$$\ln(e^x) = \ln(5)$$

On the left side, the ln and e undo each other leaving us with x, this gives the following logarithmic equation.

$$x = \ln(5)$$

Exponentiation

-We will now cover an operation called exponentiation; this is the act raising a term to the power of another term.

-The way we are going to use this is by raising a term to the power of both sides on an equation.

Convert the following logarithmic equations into exponential equations

- a.) Convert
- $\log_2(16) = 4$
- into an exponential equation.

Solution:

Since we have a log base 2, we will need an exponent with a base of 2 to undo it. We are going to exponentiate on both sides by raising 2 to the power of both sides of this equation.

$$2^{\log_2(16)} = 2^4$$

Once again, the 2 and the log base 2 are inverses and they cancel each other, therefore, all that remains on the left side of the equation is 16. Therefore, our exponential equation is as follows.

$$16 = 2^4$$

- b.) Convert
- $\log(1) = 0$
- into an exponential equation.

Solution:

Since we have a log base 10, we are going to exponentiate on both sides by raising 10 to the power of each side of this equation.

$$10^{\log(1)} = 10^0$$

The 10 and the log base 10 will cancel with one another leaving us with a 1 on the left side of the equation. Giving us the following exponential equation.

$$1 = 10^0$$

- c.) Convert
- $\log_8(2) = \frac{1}{3}$
- into an exponential equation.

Solution:

Since we have a log base 8, we are going to exponentiate on both sides by raising 8 to the power of both sides of the equation.

$$8^{\log_8(2)} = 8^{\frac{1}{3}}$$

The 8 and the log base 8 are inverses and will cancel with one another leaving us with a 2 on the left side of the equation. Therefore, our exponential equation is shown below.

$$2 = 8^{\frac{1}{3}}$$

- d.) Convert
- $\ln(e) = 1$
- into an exponential equation.

Solution:

Since we have a natural log, we are going to exponentiate on both sides by raising e to the power of both sides of the equation.

$$e^{\ln(e)} = e^1$$

The e and the ln are inverses and cancel each other, leaving us with e on the left side of the equation. That leaves us with the following exponential equation.

$$e = e^1$$

Calculating logarithms by hand

-While in most cases you will be using your calculator to calculate logarithms, it can be useful to calculate them by hand so that you can better understand not only logarithms but also, how your calculator works.

-By now, you have surely figured out that a logarithm and an exponential term with the same base will cancel one another out.

-When calculating a logarithm by hand, you must rewrite what is inside the log as a power of the same base that the logarithm has.

Example 1: Calculate the value of $\log(1000)$ by hand.

Solution:

We know that our log has a base of 10, so, we must rewrite 1000 as a power of 10.

We can clearly see that 1000 is the same as 10^3 , so, we replace it in the log.

$$\log(10^3)$$

The log base 10 and the 10 cancel out leaving 3.

Therefore, the $\log(1000) = 3$

Example 2: Calculate the value of $\log_4(1024)$ by hand.

Solution:

We know that we must rewrite 1024 as a power of 4, if we experiment with our calculator, we quickly find that 1024 is 4^5 , so, we replace it in the log.

$$\log_4(4^5)$$

The log base 4 and the 4 cancels out leaving 5.

Therefore, $\log_4(1024) = 5$

Example 3: Calculate the value of $\log_5(5)$ by hand.

Solution:

In the case, 5 is already a power of 5, it is simply 5^1 , so we can write our log as follows.

$$\log_5(5^1)$$

The log base 5 and the 5 cancels leaving us with 1.

Therefore, $\log_5(5) = 1$

Example 4: Calculate the value of $\log_3\left(\frac{1}{9}\right)$ by hand.

Solution:

This is a fun problem because you play around with it a little bit. Upon first inspection, we know that 9 is the same as 3^2 , so we replace it in the log.

$$\log_3\left(\frac{1}{3^2}\right)$$

But we have a problem, our exponent is in the denominator, but this easily be solved if we remember that $\frac{1}{3^2} = 3^{-2}$, so, we replace this in the log.

$$\log_3(3^{-2})$$

We know that the loge base 3 and the 3 can now cancel, leaving us with a -2.

$$\text{Therefore, } \log_3\left(\frac{1}{9}\right) = -2$$

Example 5: Calculate the value of $\ln\left(\frac{1}{e^7}\right)$ by hand.

Solution:

By now, you may already know that we need to remember that $\frac{1}{e^7} = e^{-7}$, so, we are going to rewrite our log expression as follows.

$$\ln(e^{-7})$$

Now, ln and e will cancel each other out and we are left with -7.

$$\text{Therefore, } \ln\left(\frac{1}{e^7}\right) = -7$$

Example 6: Calculate the value of $\log_3 \sqrt{3}$ by hand.

Solution:

We first recall that $\sqrt{3} = 3^{\frac{1}{2}}$, which means that what we really have is $\log_3 3^{\frac{1}{2}}$, it should be evident that the log base 3 and the 3 will cancel out leaving us with $\frac{1}{2}$.

$$\text{Therefore, } \log_3 \sqrt{3} = \frac{1}{2}$$

Example 7: Calculate the value of $\log_{64}(8)$ by hand.

Solution:

How can we write 8 as a power of 64? The most evident relationship between the two numbers is that $8 = \sqrt{64}$, this means that $8 = 64^{\frac{1}{2}}$. We can now rewrite our logarithm below.

$$\log_{64}(64)^{\frac{1}{2}}$$

So, the log base 64 and the 64 will cancel, leaving us with $\frac{1}{2}$.

$$\text{Therefore, } \log_{64}(8) = \frac{1}{2}$$

Example 8: Calculate the value of $\log_2(\log_3(81))$ by hand.

Solution: For space reasons, we are going to solve this without explaining each step, we will first deal with the inner log and work our way out.

$$\log_2(\log_3(81)) = \log_2(\log_3(3^4)) = \log_2(4) = \log_2(2^2) = 2$$

Example 9: Calculate the value of $\log_3(\log_7(7))$ by hand.

Solution: For space reasons, we are going to solve this without explaining each step, we will first deal with the inner log and work our way out.

$$\log_3(\log_7(7)) = \log_3(\log_7(7^1)) = \log_3(1) = 0$$