Introduction to Trigonometric Functions

-Trigonometry is one of the most useful branches of mathematics and is very applicable in science.

-Trigonometry is based on angles, distances, and triangles, specifically the relationship between an interior angle in a triangle and the ratio of its side lengths.

-Pythagorean theorem allows us to find the length of any side of a right triangle if we know the other two sides.

-Trigonometry is a more powerful tool because it allows us to find one side of a triangle based on a reference angle and 1 corresponding side.

-Trigonometry also allows us to find an interior angle based two sides of the triangle.

-Despite the diversity of the applications of trigonometric functions, the entire subject can be boiled down to 3 basic trigonometric functions and their reciprocals.

-They are sine, cosine, and tangent: each of these relate a reference angle to a ratio of sides of the same triangle.

-The mnemonic device often used to calculate them is **SOHCAHTOA.**

Sine
Opposite
Hypotenuse
Cosine
Adjacent
Hypotenuse
Tangent
Opposite
Adjacent

-This abbreviates the following:

-Sine is opposite over hypotenuse

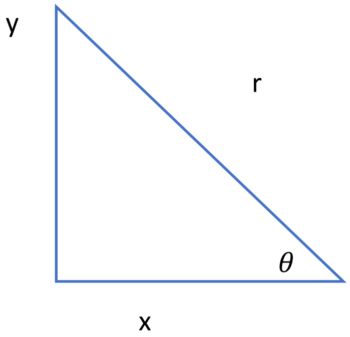
-Cosine is adjacent over hypotenuse

-Tangent is opposite over adjacent

-The sine of an angle is the ratio of the side opposite to that angle and the hypotenuse.

-The cosine of an angle is the ratio of the side adjacent to that angle and the hypotenuse.

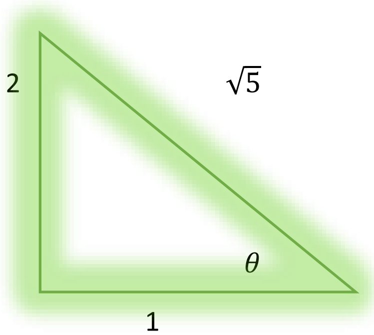
-The tangent of an angle is the ratio of the sides opposite and adjacent to the angle.



Based on the above triangle we can write the following trigonometric ratios for theta.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

Example 1: Find sine, cosine, and tangent, for the reference angle in the following triangle.



SOLUTION:

Following SOHCAHTOA we can know the following.

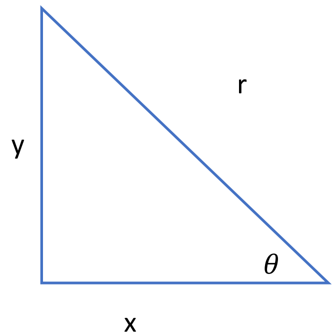
$$\sin\theta = \frac{2}{\sqrt{5}} \qquad \cos\theta = \frac{1}{\sqrt{5}} \qquad \tan\theta = \frac{2}{1}$$

-This is basically how the basic trig functions work, each of them relates an angle to a ratio.

-Each of the 3 trigonometric functions have reciprocal functions that are detailed in the table below.

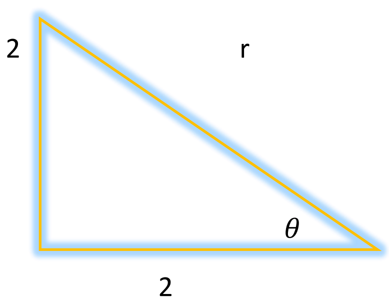
The Six Trigonometric Functions

Below are the six trigonometric functions as they correspond to the given triangle.



Sine	$\boldsymbol{\sin\theta = \frac{y}{r}}$	**→ Ratio →**	Cosecant	$\boldsymbol{\csc\theta = \frac{r}{y}}$
Cosine	$\cos\theta = \frac{x}{r}$	→ Ratio →	Secant	$\sec\theta = \frac{r}{x}$
Tangent	$\tan\theta = \frac{y}{x}$	→ Ratio →	Cotangent	$\cot\theta = \frac{x}{y}$

Example 2: Find all 6 trigonometric functions for θ in the following triangle.



SOLUTION:

In this problem, we are missing the hypotenuse of the triangle, so, the first thing that we will do is find the hypotenuse using Pythagorean theorem.

$$2^2 + 2^2 = r^2$$

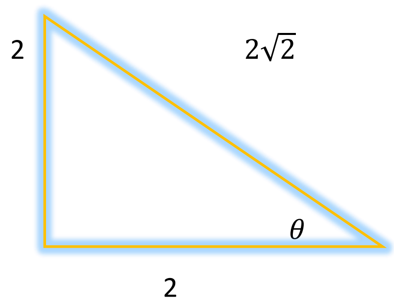
$$4 + 4 = r^2$$

$$8 = r^2$$

$$\sqrt{8} = \sqrt{r^2}$$

$$2\sqrt{2} = r$$

Now that we now the value of x, our triangle actually looks like the triangle below.



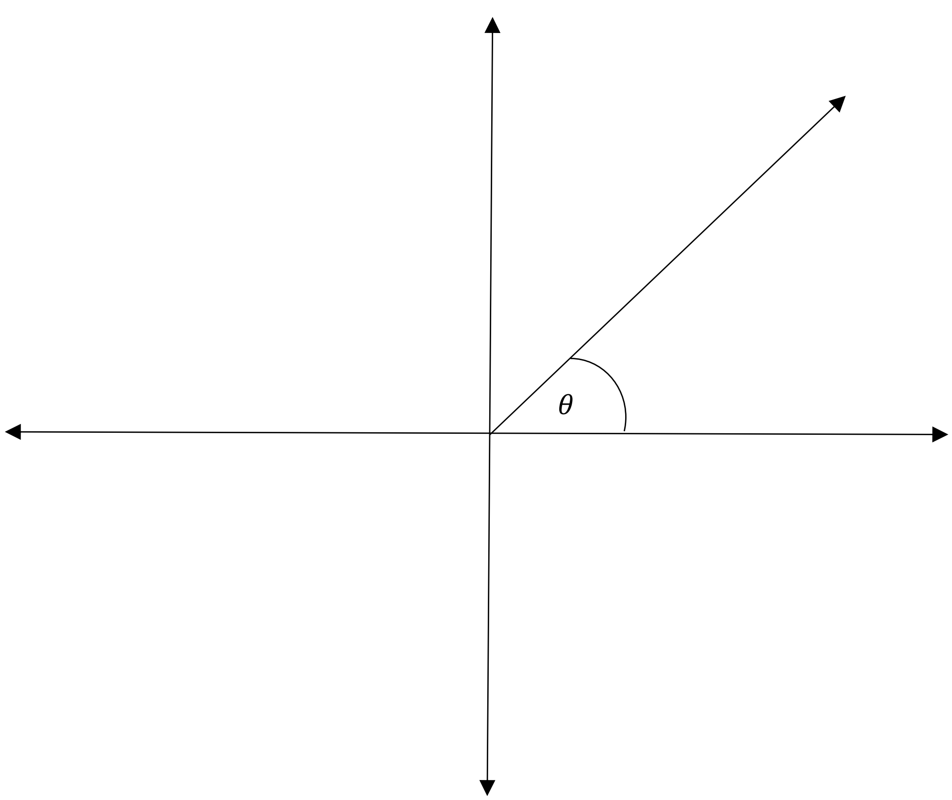
Now, we can apply SOHCAHTOA to find the 6 trigonometric ratios for the angle.

$$\sin\theta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \cos\theta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \tan\theta = \frac{2}{2} = 1$$

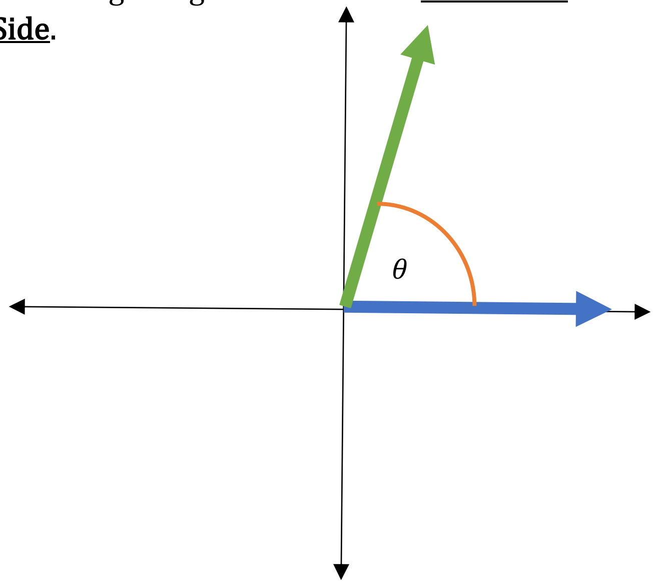
$$\csc\theta = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \sec\theta = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \cot\theta = \frac{1}{1} = 1$$

Trigonometry and Angles on the Cartesian Plane

-**Standard Position**- An angle that is rising from the positive x-axis.



-The side where an angle begins is called the **<u>Initial Side</u>** and the side where it ends is called the **<u>Terminal Side</u>**.

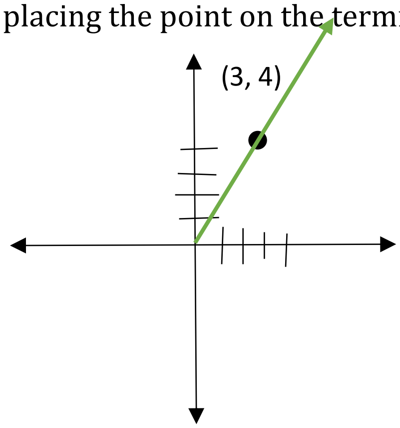


-In the above diagram, the green side is the Terminal Side and the blue side is the Initial Side.

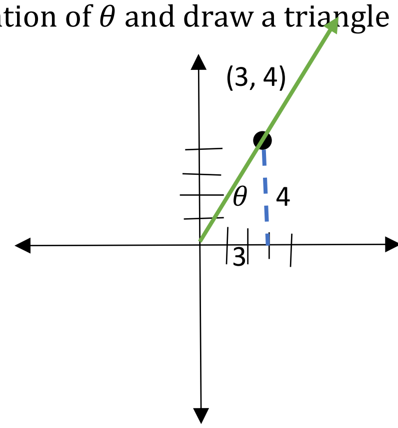
Example 3: Find all six trigonometric functions for θ if $(3, 4)$ is on the terminal side of θ.

SOLUTION:

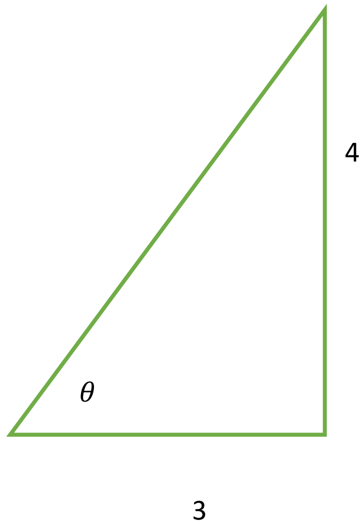
First, we sketch the angle placing the point on the terminal side of the angle.



-Now, we can see the location of θ and draw a triangle as shown below.



-What we really have is the following triangle.



We need Pythagorean Theorem to find the value of the hypotenuse which we will do below.

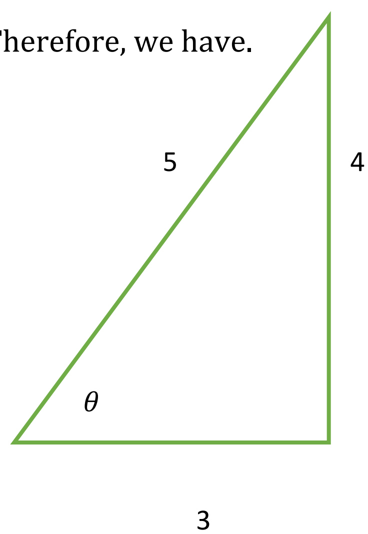
$$3^2 + 4^2 = r^2$$

$$9 + 16 = r^2$$

$$25 = r^2$$

$$5 = r$$

Therefore, we have.



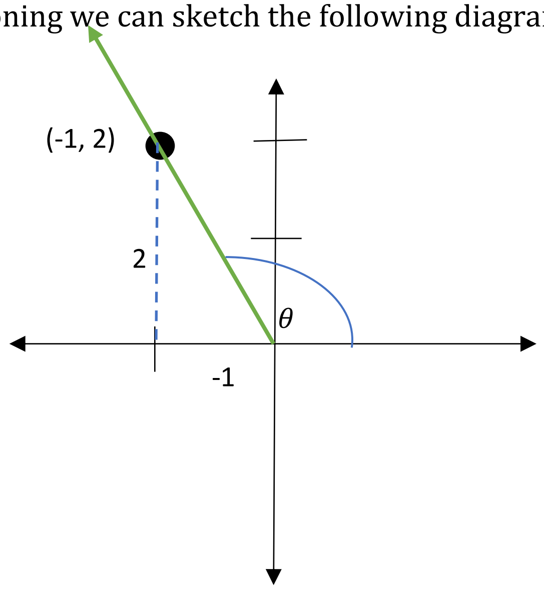
Now that the triangle is complete, we can state our trig functions according to SOHCAHTOA below.

$\mathbf{\sin\theta = \frac{4}{5}}$	$\mathbf{\cos\theta = \frac{3}{5}}$	$\mathbf{\tan\theta = \frac{4}{3}}$
$\mathbf{\csc\theta = \frac{5}{4}}$	$\sec\theta = \frac{5}{3}$	$\cot\theta = \frac{3}{4}$

Example 4: Find all six trigonometric functions for θ if $(-1, 2)$ is on the terminal side of θ.

SOLUTION:

Using similar reasoning we can sketch the following diagram.



-We plotted the point and drew the angle; we can see that the two legs are 2 and -1.

-You may wonder how -1 can possibly be a length, the reason that we can use it in this instance is that the -1 is not a length. Instead, -1 is a location, and it and along with the y coordinate tells us that we are in the second quadrant.

-Next we will find the hypotenuse using Pythagorean theorem.

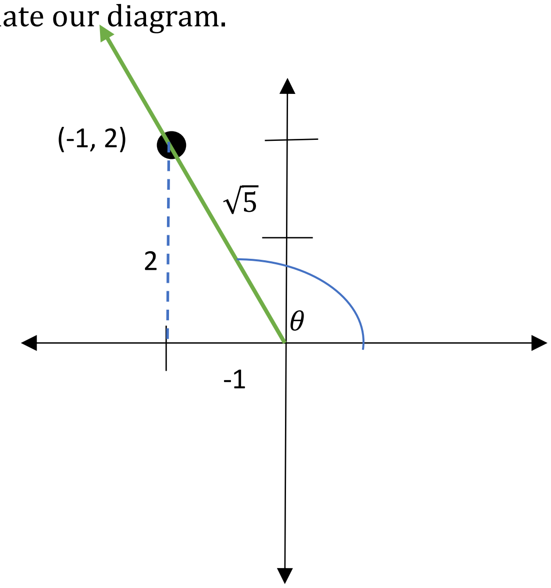
$$(2)^2 + (-1)^2 = r^2$$

$$4 + 1 = r^2$$

$$5 = r^2$$

$$\sqrt{5} = r$$

Therefore, we update our diagram.



-Since our triangle is complete, we can write our six trig functions below.

$\sin\theta = \frac{2}{\sqrt{5}}$	$\cos\theta = -\frac{1}{\sqrt{5}}$	$\tan\theta = \frac{2}{-1} = -2$
$\csc\theta = \frac{\sqrt{5}}{2}$	$\sec\theta = -\sqrt{5}$	$\cot\theta = -\frac{1}{2}$

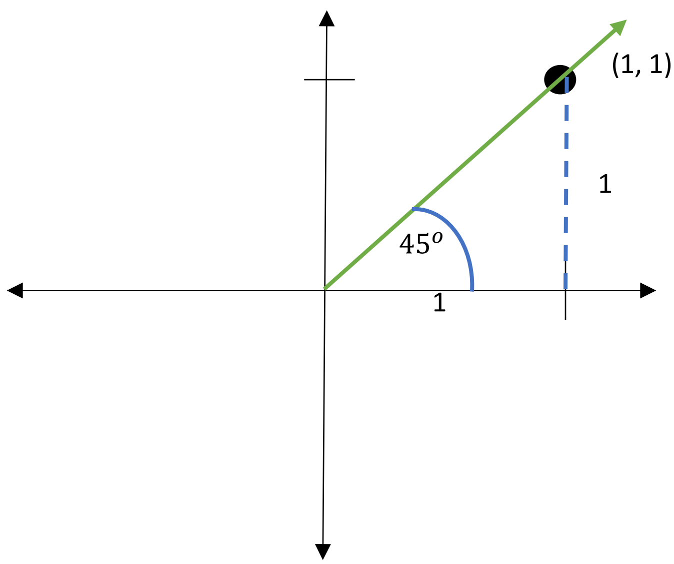
Example 5: Find the sine, cosine, and tangent for 45^o.

SOLUTION:

This is an interesting question because 45^o is an interesting angle. It is in the first quadrant and more importantly its terminal side is the diagonal that cuts the first quadrant exactly in half.

Therefore, points on this terminal side are all point in Q1 where $y = x$. This means that points such as $(1, 1), (2, 2), (3, 3), (4, 4)$ ... and so on, are all on the terminal side of a 45^o angle.

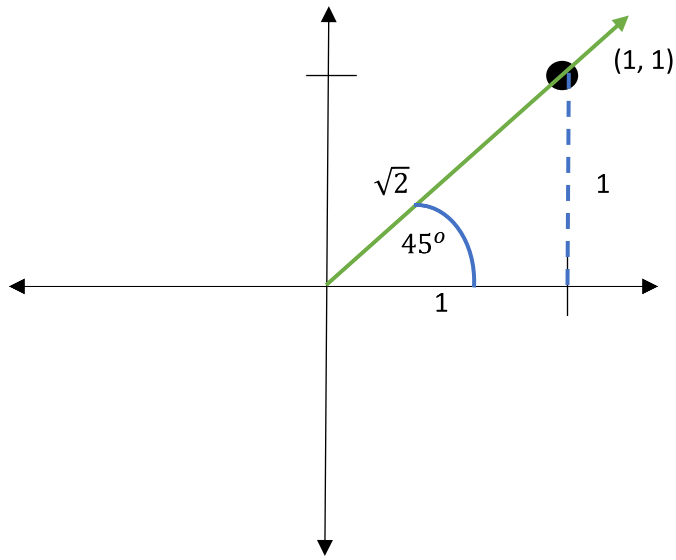
Any of these points will work, however, since we like simplicity, we will choose $(1, 1)$ and we can draw the following diagram.



We have two sides of the triangle and we can use Pythagorean theorem to find the hypotenuse.

$$\mathbf{1^2 + 1^2 = r^2}$$
$$\mathbf{1 + 1 = r^2}$$
$$\mathbf{2 = r^2}$$
$$\mathbf{\sqrt{2} = r}$$

Now that we know the hypotenuse, we can update our diagram.

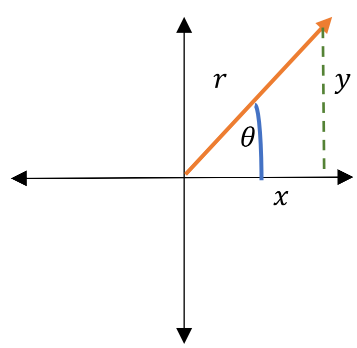


Now, we can apply SOHCAHTOA to find the sine, cosine, and tangent of 45^o below.

$$\sin(45^o) = \frac{1}{\sqrt{2}} \qquad \cos(45^o) = \frac{1}{\sqrt{2}} \qquad \tan(45^o) = \frac{1}{1} = 1$$

GENERAL RULES ABOUT TRIGONOMETRIC FUNCTIONS

-After what we have seen about trigonometric functions, we can draw the following diagram to demonstrate the rules.



Based on this diagram, we can write the following trigonometric functions about θ.

$\boldsymbol{\sin\theta = \frac{y}{r}}$	$\boldsymbol{\cos\theta = \frac{x}{r}}$	$\boldsymbol{\tan\theta = \frac{y}{x}}$
$\boldsymbol{\csc\theta = \frac{r}{y}}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{x}{y}$

The above table encapsulates three very important points.

-Sine has to do with the y-coordinate

-Cosine has to do with the x-coordinate

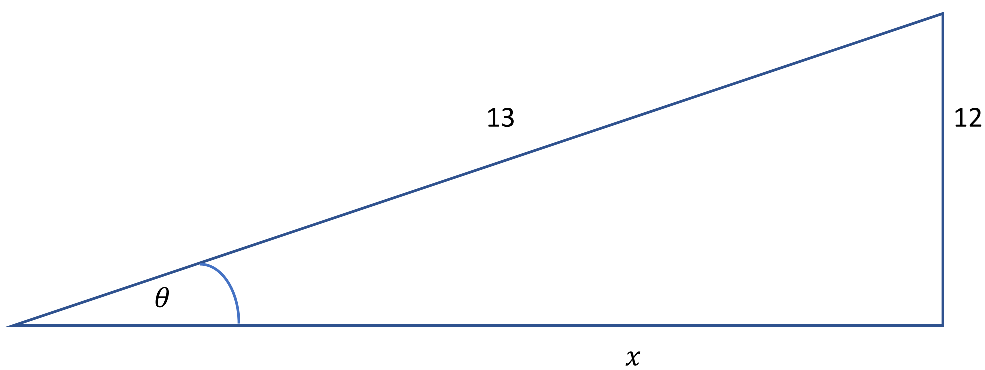
-Tangent is the y-coordinate divided by the x coordinate.

Knowing these simple facts will be very beneficial for solving problem like the one we show below.

Example 6: Find the remaining 5 trigonometric functions if $\sin\theta = \frac{12}{13}$ and θ terminates in Q2.

SOLUTION:

We start by remembering that $\sin\theta = \frac{opposite}{hypotenuse}$ which means that 12 is opposite the angle and 13 is the hypotenuse. We can draw the following triangle.



We are missing a side and so we will employ Pythagorean theorem to find that side.

$$x^2 + 12^2 = 13^2$$

$$x^2 + 144 = 169$$

$$x^2 = 25$$

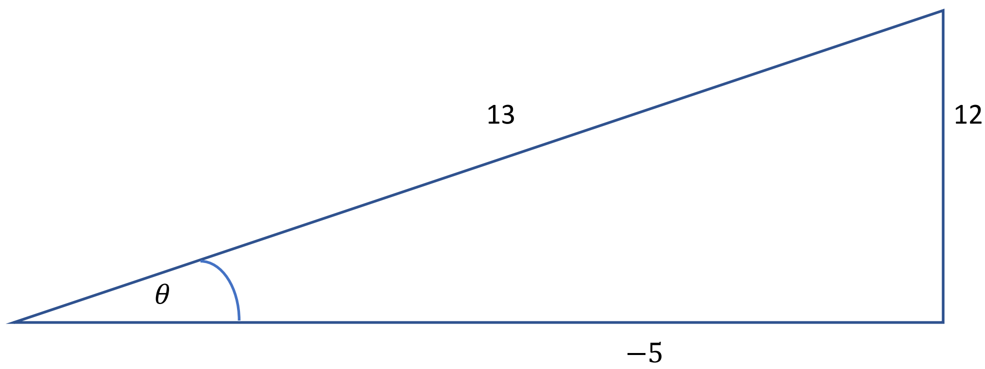
$$\sqrt{x^2} = \sqrt{25}$$

$$x = \pm 5$$

Ordinarily, at this stage, we would choose the positive value because a length cannot be negative, but remember, the negative no longer corresponds to a length instead it tells us a location.

We know that our angle terminates in the second quadrant and we know that x is negative in the second quadrant, therefore, we choose the negative in this case.

So, we can rewrite our diagram as follows.



Now that we know all the sides, we can state all of the trigonometric ratios below.

$$\sin\theta = \frac{12}{13} \qquad \cos\theta = -\frac{5}{13} \qquad \tan\theta = -\frac{12}{5}$$

$$\csc\theta = \frac{13}{12} \qquad \sec\theta = -\frac{13}{5} \qquad \cot\theta = -\frac{5}{12}$$

Important Note:

-Please remember that the hypotenuse is always positive not matter what quadrant it is located in.