Partial Fraction Decomposition

-Partial Fraction Decomposition as its name states, is the operation of decomposing a fractional expression into its separate fraction parts.

-You may remember that as an Algebra student you had to do problems like the following example.

$$\frac{2}{x+1} + \frac{3}{x-2} \rightarrow \left(\frac{x-2}{x-2}\right)\frac{2}{x+1} + \left(\frac{x+1}{x+1}\right)\frac{3}{x-2}$$

$$\frac{2x-4}{(x+1)(x-2)} + \frac{3x+3}{(x+1)(x-2)} = \frac{5x-1}{(x+1)(x-2)} = \frac{5x-1}{x^2-x-2}$$

-The act of going backwards from $\frac{5x-1}{x^2-x-2}$ to $\frac{2}{x+1} + \frac{3}{x-2}$ is called a Partial Fraction Decomposition.

-This is a very useful skill that comes in handy in Calculus.

-It also happens to be a great application of systems of equations.

Steps for performing Partial Fraction Decomposition

1.) Make sure that the denominator is factored.
2.) Identify the form of the decomposition.
3.) Set the expression equal to its decomposed form.
4.) Clear the fractions by multiplying by the expression's denominator.
5.) Completely FOIL and distribute the right ride.
6.) Group like terms together in the right side.
7.) Factor the variable out of each group if possible.
8.) Split equations by setting each group equal to its counterpart.
9.) Solve the resulting system of equations.
10.) Once you have A, B, C, and so on, you can state your decomposition.

Partial Fraction Decomposition with Distinct Linear Factors

-For a term where the denominator has distinct linear factors, the partial fraction decomposition will have the following general form.

$$\frac{P(x)}{(ax+b)(cx+d)(ex+f)\ldots} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f} + \cdots$$

-This may seem confusing but hopefully if we put it into practice it will make more sense.

Example 1: Find the partial fraction decomposition for the following rational expression.

$$\frac{3x+50}{(x-9)(x+2)}$$

SOLUTION:

First, since the denominator is already factored, we don't need to do that.

Second, since the denominator of this expression contains two distinct linear factors the form of our decomposition will be as follows.

$$\frac{A}{x-9}+\frac{B}{x+2}$$

Third, will set this equal to our expression, the goal is to find the values of A and B which will make this expression true.

$$\frac{3x+50}{(x-9)(x+2)}=\frac{A}{x-9}+\frac{B}{x+2}$$

Fourth, we will clear the fractions by multiplying by the denominator of our original expression.

$$(x-9)(x+2)\left(\frac{3x+50}{(x-9)(x+2)}=\frac{A}{x-9}+\frac{B}{x+2}\right)$$

This will simplify to give us the following expression.

$$3x+50=A(x+2)+B(x-9)$$

Fifth, we will distribute A and B.

$$3x+50=Ax+2A+Bx-9B$$

Sixth, we will group the x terms and the constant terms on the right side.

$$3x+50=Ax+AB+2A-9B$$

Seventh, we will factor (when possible) the variables out.

$$3x+50=x(A+B)+(2A-9B)$$

Eighth, the above equation is color coded to show that the x's equal the x's and the constants equal the constants, in other words, the blues equal the blues and the greens equal the greens. We can now split into the following system of equations.

$$\begin{cases} A+B=3 \\ 2A-9B=50 \end{cases}$$

Ninth, we need to solve this system of equations. This can be done using substitution, elimination, or matrices. To save time, we will use and matrix and the TI-84 to solve this system.

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -9 & 50 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \end{bmatrix}$$

This means that A=7 and B=-4.

Tenth, now that we know A and B, we plug those into our decomposition for the final answer.

$$\frac{7}{x-9} + \frac{-4}{x+2} \quad or \quad \frac{7}{x-9} - \frac{4}{x+2}$$

Example 2: Find the Partial Fraction Decomposition for the following rational expression.

$$\frac{x}{x^2 - 5x + 6}$$

SOLUTION:

First, we factor the denominator.

$$\frac{x}{(x-3)(x-2)}$$

Second, since this expression has distinct linear terms, the form of the decomposition can be seen below.

$$\frac{A}{x-3} + \frac{B}{x-2}$$

Third, we set this equal to the original expression.

$$\frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

Fourth, we clear the fractions.

$$(x-3)(x-2)\left(\frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}\right)$$

This will simplify to the equation below.

$$x = A(x-2) + B(x-3)$$

Fifth, we will distribute.

$$x = Ax - 2A + Bx - 3B$$

Sixth, we will group the x terms and the constant terms on the right side.

$$x = Ax + Bx - 2A - 3B$$

Seventh, we will factor x out where it is possible.

$$1x = x(A + B) - 2A - 3B$$

Eighth, the blues equal the blues, but there is no red constant on the left side to equal the red constant on the right, therefore, you set it equal to 0. This gives us the following system of equations.

$$\begin{cases} A + \quad B = 1 \\ -2A - 3B = 0 \end{cases}$$

Ninth, we will solve this use a matrix and calculator as in the previous problem.

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

This means that A=3 and B=-2.

Tenth, now that we know A and B, we plug those into our decomposition for the final answer.

$$\frac{3}{x-3} + \frac{-2}{x-2} \quad or \quad \frac{3}{x-3} - \frac{2}{x-2}$$

Example 3: Find the Partial Fraction Decomposition for the following rational expression.

$$\frac{2x^2 - 18x - 12}{x^3 - 4x}$$

SOLUTION:

First, we factor the denominator.

$$x^3 - 4x = x(x^2 - 4) = x(x + 2)(x - 2)$$

This means that we can write our rational expression as follows.

$$\frac{2x^2 - 18x - 12}{x(x + 2)(x - 2)}$$

Second, since we have 3 distinct linear factors, that means that the form of our decomposition will be as follows.

$$\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

Third, we set our rational expression equation to its decomposed form.

$$\frac{2x^2 - 18x - 12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

Fourth, we will clear the fraction by multiplying by the denominator on the left side.

$$x(x+2)(x-2)\left(\frac{2x^2 - 18x - 12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}\right)$$

This will simplify into the following equation.

$$2x^2 - 18x - 12 = A(x+2)(x-2) + B\big(x(x-2)\big) + C\big(x(x+2)\big)$$

Fifth, we FOIL and distribute all x terms on the left side.

$$2x^2 - 18x - 12 = A(x^2 - 4) + B(x^2 - 2x) + C(x^2 + 2x)$$

Lastly, we will distribute the A, B, and C.

$$2x^2 - 18x - 12 = Ax^2 - 4A + Bx^2 - 2Bx + Cx^2 + 2Cx$$

Sixth, we group the x^2, x, and constant terms on the right side.

$$2x^2 - 18x - 12 = Ax^2 + Bx^2 + Cx^2 - 2Bx + 2Cx - 4A$$

Seventh, we factor the variables out when possible.

$$2x^2 - 18x - 12 = x^2(A + B + C) + x(-2B + 2C) + (-4A)$$

Eighth, we split out equation up to get the following system.

$$\begin{cases} A + B + C = 2 \\ -2B + 2C = -18 \\ -4A = -12 \end{cases}$$

Ninth, we will solve the system of equations using a matrix and the calculator.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & -18 \\ -4 & 0 & 0 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

This means that A=3, B=4, and C=-5

Tenth, therefore, our decomposition is as follows.

$$\frac{3}{x} + \frac{4}{x+2} - \frac{5}{x-2}$$

Partial Fraction Decomposition with Repeated Linear Factors

-In this case we have linear factors in the denominator that are not distinct and are therefore repeated.

-The set up is a little different in this case and it follows the pattern below.

$$\frac{P(x)}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{Z}{(ax+b)^n}$$

Example 1: Find the partial fraction decomposition for the following rational expression.

$$\frac{6x-11}{(x-1)^2}$$

First, the denominator is already factored so we can move on.

Second, since we have a linear factor that is repeated twice in the denominator, our decomposition will have the following form.

$$\frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Third, we set the expression equal to its decomposed form to find A and B.

$$\frac{6x-11}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Fourth, we will clear the fractions.

$$(x-1)^2\left(\frac{6x-11}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}\right)$$

This will simplify as follows.

$$6x - 11 = A(x-1) + B$$

Fifth, we will distribute on the right side of the equation.

$$6x - 11 = Ax - A + B$$

Sixth, things are actually already grouped for us.

$$6x - 11 = Ax - A + B$$

Seventh, we don't need to pull x out in this case.

Eighth, we get the following system.

$$\begin{cases} A = 6 \\ -A + B = -11 \end{cases}$$

Ninth, while this is a very simple system to solve, we will go ahead and use a matrix for the sake of uniformity.

$$\begin{bmatrix} 1 & 0 & 6 \\ -1 & 1 & -11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -5 \end{bmatrix}$$

Therefore, A=6 and B=-5.

Tenth, the partial fraction decomposition is as follows.

$$\frac{6}{x-1} - \frac{5}{(x-1)^2}$$

Example 2: Find the partial fraction decomposition for the following rational expression.

$$\frac{x^2+2x+7}{x(x-1)^2}$$

SOLUTION:

First, the denominator is already factored, so, we can move on.

Second, since we have one distinct linear factor and two repeated linear factors, out decomposition will have the following form.

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Third, we set our rational expression equal to its decomposition.

$$\frac{x^2+2x+7}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Fourth, we clear all fractions.

$$x(x-1)^2\left(\frac{x^2+2x+7}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right)$$

This will simplify to the following equation.

$$x^2+2x+7 = A(x-1)^2 + Bx(x+1) + Cx$$

Fifth, we FOIL and distribute on the right side of the equation.

$$x^2+2x+7 = A(x^2-2x+1) + B(x^2+x) + Cx$$

$$x^2+2x+7 = Ax^2 - 2Ax + A + Bx^2 + Bx + Cx$$

Sixth, we will group like terms on the right side.

$$x^2+2x+7 = Ax^2 + Bx^2 - 2Ax + Bx + Cx + A$$

Seventh, we will factor the variables out when possible.

$$1x^2 + 2x + 7 = x^2(A + B) + x(-2A + B + C) + A$$

Eighth, we can now split this equation into the following system of equation.

$$\begin{cases} A + B = 1 \\ -2A + B + C = 2 \\ A = 7 \end{cases}$$

Ninth, we will once again solve this system using a matrix and the calculator.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 0 & 0 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 22 \end{bmatrix}$$

We can see that our solutions are A=7, B=-6, and C=22.

Tenth, this means that our partial fraction decomposition is as follows.

$$\frac{7}{x} - \frac{6}{x-1} + \frac{22}{(x-1)^2}$$

Partial Fraction Decomposition with Non-repeated Prime Quadratic Factors

-When you have a quadratic factor in the denominator that cannot be factored, it is called prime.

-In these cases, your decomposition will contain a term that looks like the following.

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example 1: Find the partial fraction decomposition for the following rational expression.

$$\frac{5x^2 + 6x + 3}{(x+1)(x^2 + 2x + 2)}$$

SOLUTION:

First, we see that the denominator is fully factored.

Second, we see that our denominator contains one distinct linear factor and one distinct prime quadratic factor, because of this, my decomposition will have the following form.

$$\frac{A}{x+1} + \frac{Bx + C}{x^2 + 2x + 2}$$

Third, we set our rational expression equal to decomposition.

$$\frac{5x^2 + 6x + 3}{(x + 1)(x^2 + 2x + 2)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 2x + 2}$$

Fourth, we will clear all fractions.

$$(x + 1)(x^2 + 2x + 2)\left(\frac{5x^2 + 6x + 3}{(x + 1)(x^2 + 2x + 2)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 2x + 2}\right)$$

This will simplify to the equation below.

$$5x^2 + 6x + 3 = A(x^2 + 2x + 2) + (Bx + C)(x + 1)$$

Fifth, we will FOIL and distribute to simplify the expression.

$$5x^2 + 6x + 3 = Ax^2 + 2Ax + 2A + Bx^2 + Bx + Cx + C$$

Sixth, we will group all like terms in the right side.

$$5x^2 + 6x + 3 = Ax^2 + Bx^2 + 2Ax + Bx + Cx + 2A + C$$

Seventh, we will factor out variables whenever possible.

$$5x^2 + 6x + 3 = x^2(A + B) + x(2A + B + C) + (2A + C)$$

Eighth, we will split this up into the following system of equations.

$$\begin{cases} A + B = 5 \\ 2A + B + C = 6 \\ 2A + C = 3 \end{cases}$$

Ninth, we enter this into the following matric and solve it in the calculator.

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 2 & 1 & 1 & 6 \\ 2 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We can see that A=2, B=3, and C=-1

Tenth, since we now know A, B, and C, our decomposition can be written below.

$$\frac{2}{x + 1} + \frac{3x - 1}{x^2 + 2x + 2}$$

Partial Fractions Decomposition with Repeated Prime Quadratic Factors

-When the denominator of your rational expression contains repeated prime quadratic factors, your decomposition will have the following form.

$$\frac{P(x)}{(ax^2+bx+c)^n}=\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}+\cdots+\frac{Yx+Z}{(ax^2+bx+c)^n}$$

Example 1: Find the partial fraction decomposition for the following rational expression.

$$\frac{x^3-4x^2+9x-5}{(x^2+2)^2}$$

SOLUTION:

First, we can see that the denominator cannot be factored anymore, so we move on.

Second, since the denominator contains 2 repeated prime quadratic factors, our decomposition will have the following form.

$$\frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}$$

Third, we set this equal to our rational expression.

$$\frac{x^3-4x^2+9x-5}{(x^2+2)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}$$

Fourth, we clear all fractions.

$$(x^2+2)^2\left(\frac{x^3-4x^2+9x-5}{(x^2+2)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}\right)$$

This will simplify as follows.

$$x^3-4x^2+9x-5=(Ax+B)(x^2+2)+Cx+D$$

Fifth, we will FOIL and distribute to simplify the right side of the equation.

$$x^3-4x^2+9x-5=Ax^3+2Ax+Bx^2+2B+Cx+D$$

Sixth, we group like terms on the right side.

$$x^3-4x^2+9x-5=Ax^3+Bx^2+2Ax+Cx+2B+D$$

Seventh, we factor where possible.

$$1x^3 - 4x^2 + 9x - 5 = (A)x^3 + (B)x^2 + x(2A + C) + (2B + D)$$

Eight, we can now split this into the following system of equations.

$$\begin{cases} A = 1 \\ B = -4 \\ 2A + C = 9 \\ 2B + D = -5 \end{cases}$$

Ninth, since we get two solutions for free, substitution would really be the easiest way to solve this system, however, for the sake of uniformity, we will use a matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -4 \\ 2 & 0 & 1 & 0 & 9 \\ 0 & 2 & 0 & 1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

We can see that our solutions at A=1, B=-4, C=7, and D=3

Tenth, since we know A, B, C, and D, we can write our Partial Fraction Decomposition below.

$$\frac{x-4}{x^2+2} + \frac{7x+3}{(x^2+2)^2}$$