Properties of Logarithms

-You have probably figured out by now that logarithms are actually exponents!

-Due to this, they possess some unique properties that make them even more useful.

-In this tutorial we will cover the properties of logarithms and use them to perform expansions and contractions.

Property 1: The Power Rule

$$\log_b x^y = y \log_b x$$

-As you can see, this rule allows us to pull the exponent down from of the log.

-Lets' apply the power rule in some examples as seen below.

a.) $\log x^2 = 2 \log x$

b.) $\ln(x+1)^3 = 3 \ln(x+1)$

c.) $\log_2 \sqrt{x} = \frac{1}{2} \log_2(x)$

Property 2: The Product Rule

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

-As you can see, the product rule allows us to pull products apart with addition.

-Lets' apply the product rule in some examples as soon below.

a.) $\log(3x) = \log(3) + \log(x)$

b.) $\ln(x(x+1)) = \ln(x) + \ln(x+1)$

c.) $\log_5(3xy) = \log_5(3) + \log_5(x) + \log_5(y)$

Property 3: The Quotient Rule

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

-As you can see, the product rule allows us to pull fractions apart with subtraction.

-Lets' apply the product rule in some examples below.

a.) $\ln\left(\frac{x+2}{3-x}\right) = \ln(x+2) - \ln(3-x)$

b.) $\log_2\left(\frac{3}{xy}\right) = \log_2(3) - \log_2(xy)$

c.) $\log\left(\frac{x(x+2)}{x^2-1}\right) = \log(x(x+2) - \log(x^2-1)$

Logarithmic Expansions

-Now that we have covered the basic rules of logarithms, we can combine to perform what is called a logarithmic expansion.

-A logarithmic expansion is when you use the rules of logarithms to pull a complicated logarithmic expression apart until you have a series of separate simple logarithmic expression.

-This is a useful trick that is often used in Calculus for what is called logarithmic differentiation.

-We will start with simple examples and work up to more complicated ones.

Example 1: Use the properties of logarithms to expand the following logarithmic expression.

$$\log_5\left(\frac{x(x+2)}{y^2}\right)$$

SOLUTION:

-The first thing that we want to break up in this case is the fraction, we will do this using the quotient rule.

$$\log_5(x(x+2)) - \log_5(y^2)$$

-We see that the first log contains a product so we will next split that up using the product rule.

$$\log_5(x) + \log_5(x+2) - \log_5(y^2)$$

-Lastly, we see that the last log contains a power that can be removed using the power rule.

$$\log_5(x) + \log_5(x+2) - 2\log_5(y)$$

-Now, our logarithm is fully expanded.

Example 2: Use the properties of logarithms to expand the following logarithmic expression.

$$\ln\left[\frac{xe^5}{x\sqrt{x-2}}\right]$$

SOLUTION:

-As in the last problem, we will start with the fraction and break it up using the quotient rule.

$$\ln(xe^5) - \ln(x\sqrt{x-2})$$

-Next, we notice that each of our logarithms contains a product that can be split using the product rule.

$$\ln(x) + \ln(e^5) - [\ln(x) + \ln\sqrt{x-2}]$$

-Now, we will distribute the negative to obtain.

$$\ln x + \ln e^5 - \ln x - \ln\sqrt{x-2}$$

-We can see that the two terms (ln x) will cancel leaving us with.

$$\ln e^5 - \ln\sqrt{x-2}$$

-In our second logarithm, we see a square root, but know that that is really an exponent with a ½ power, so, we can pull that out using the power rule.

$$\ln e^5 - \frac{1}{2}\ln(x-2)$$

-While it is tempting to think that this if over, we remember that the ln and e are inverses that cancel each other out leaving us with 5, leaving us with a final answer of.

$$5 - \frac{1}{2}\ln(x-2)$$

Example 3: Use the properties of logarithms to expand the following logarithmic expression.

$$\log\left[\frac{10x\sqrt[3]{2x+3}}{3(3x-1)^5}\right]$$

SOLUTION:

-First, we are going to break the fraction using the quotient rule.

$$\log[10x\sqrt[3]{2x+3}] - \log[3(3x-1)^5]$$

-Our first log contains 3 products and our second log contains 2 so we are going to split these up using the product rule.

$$\log 10 + \log x + \log\sqrt[3]{2x+3} - [\log 3 + \log(3x-1)^5]$$

-Next, we will distribute the negative.

$$\log 10 + \log x + \log\sqrt[3]{2x+3} - \log 3 - \log(3x-1)^5$$

-We have two logs that have powers in them, so, we will use the power rule next.

$$\log 10 + \log x + \frac{1}{3}\log(2x + 3) - \log 3 - 5\log(3x - 1)$$

-Lastly, since our logarithm has a base of 10, we know that when we put 10 inside it, it reduces to 1, therefore we get the following fully expanded logarithm.

$$1 + \log x + \frac{1}{3}\log(2x + 3) - \log 3 - 5\log(3x - 1)$$

Example 4: Use the properties of logarithms to expand the following logarithmic expression.

$$\log_2\left[16x^2\sqrt{\frac{2x-1}{x+4}}\right]$$

SOLUTION:

-This problem is little more entertaining than previous problems and while we usually start with the fraction, that won't work in this case, instead we note that inside our log, we have a product of 3 terms, we will separate them using the product rule.

$$\log_2 16 + \log_2 x^2 + \log_2\sqrt{\frac{2x-1}{x+4}}$$

-Now, we will apply the power rule to pull each power our front.

$$\log_2 16 + 2\log_2 x + \frac{1}{2}\log_2\left(\frac{2x-1}{x+4}\right)$$

-Next, we will apply the quotient rule to split the fraction within the last log apart.

$$\log_2 16 + 2\log_2 x + \frac{1}{2}(\log_2(2x-1) - \log_2(x+4))$$

-We distribute the ½.

$$\log_2 16 + 2\log_2 x + \frac{1}{2}\log_2(2x-1) - \frac{1}{2}\log_2(x+4)$$

-While it may be tempting to say that we are done at this point, it turns out that there is one more thing that will simplify, $\log_2 16 = 4$, so this gives a final answer of.

$$4 + 2\log_2 x + \frac{1}{2}\log_2(2x-1) - \frac{1}{2}\log_2(x+4)$$

Logarithmic Contractions

-Contracting is the process of taking series of logarithms that are being added or subtracted and combining them into one logarithm by reversing the rules.

-We often use this process to make solution look neater and more organized in classes like calculus.

-We will literally use the reverse of the product, power, and quotient rules.

Example 1: Contract the following logarithmic expression.

$$2\ln(x+1)+\ln 2-3\ln(x)-\ln(x-1)$$

SOLUTION:

-First, we can see some coefficients in front of the logs, and we know from the power rule that we can pull those up as exponents, so first we will do that.

$$\ln(x+1)^2+\ln 2-\ln x^3-\ln(x-1)$$

-The first two logs are being added and therefore that means that we can reverse the product rule and combine them as follows.

$$\ln(2(x+1)^2)-\ln x^3-\ln(x-1)$$

-Next, notice that the two last logs are both negative, so we can factor a -1 out. (In general, when two logs have the same coefficient, you want to factor it out.)

$$\ln(2(x+1)^2)-(\ln x^3+\ln(x-1))$$

-Now we can see that there is a $+$ between the last two logs and we can combine them by reversing the product rule.

$$\ln(2(x+1)^2)-\ln(x^3(x-1))$$

-Since these two logs are being subtracted, we can reverse the quotient rule to get the following solution.

$$\ln\left[\frac{2(x+1)^2}{x^3(x-1)}\right]$$

Example 2: Contract the following logarithmic expression.

$$\frac{1}{2}\log(x+1)+\frac{1}{2}\log(3x-7)-\frac{1}{2}\log(2x-1)-\frac{1}{2}\log(2x-3)$$

SOLUTION:

-The first thing that jumps out at us is the fact that all terms share a coefficient of 1/2, so we are going to factor that out of everything.

$$\frac{1}{2}[\log(x+1)+\log(3x-7)-\log(2x-1)-\log(2x-3)]$$

-Next we notice that the first two logs have a + between them which means that we can combine them by reversing the product rule.

$$\frac{1}{2}[\log[(x+1)(3x-7)]-\log(2x-1)-\log(2x-3)]$$

-Notice that the last two logs are both negative, therefore we want to factor the negative out.

$$\frac{1}{2}[\log[(x+1)(3x-7)]-(\log(2x-1)+\log(2x-3))]$$

-Now we can combine the last two logs by reversing the production rule.

$$\frac{1}{2}[\log[(x+1)(3x-7)]-\log[(2x-1)(2x-3)]$$

-Since there is a minus sign between these two logs, we can combine them by reversing the quotient rule.

$$\frac{1}{2}\log\left[\frac{(x+1)(3x-7)}{(2x-1))(2x-3)}\right]$$

-Lastly, the ½ can be pulled up as a square root by reversing the power rule giving us the following final solution.

$$\log\sqrt{\frac{(x+1)(3x-7)}{(2x-1)(2x-3)}}$$

Example 3: Contract the following logarithmic expression.

$$\log_3 5+\frac{1}{2}\log_3(x^2-3x+1)-\log_3 7-2\log_3 x-5\log_3(7x-2)$$

SOLUTION:

-To start with, we have 3 coefficients (1/2, 2, and 5) that can all be pulled up reversing the power rule.

$$\log_3 5 + \log_3 \sqrt{x^2 - 3x + 1} - \log_3 7 - \log_3 x^2 - \log_3 (7x - 2)^5$$

-The first two logs have a + between them so we can combine them by reversing the product rule.

$$\log_3 (5\sqrt{x^2 - 3x + 1}) - \log_3 7 - \log_3 x^2 - \log_3 (7x - 2)^5$$

-Next, we see that the 3 last logarithms are all negative therefore we can pull a negative out from all three of them.

$$\log_3 (5\sqrt{x^2 - 3x + 1}) - [\log_3 7 + \log_3 x^2 + \log_3 (7x - 2)^5]$$

-We can combine the 3 logarithms in the brackets by reversing the product rule.

$$\log_3 (5\sqrt{x^2 - 3x + 1}) - \log_3 (7x^2 (7x - 2)^5)$$

-Lastly, we can reverse the quotient rule to combine these logs into one.

$$\log_3 \left[\frac{5\sqrt{x^2 - 3x + 1}}{7x^2 (7x - 2)^5}\right]$$