Proving Trigonometric Identities

-Among the common precalculus topics, proving identities is often considered to be the most difficult of topics.

-What many students find confusing is that fact that there are no cookie cutter methods for proving a trig identity, interesting this is what makes them such fun.

-You can usually prove an identity several different ways, and they are all correct.

-The goal is to take one side of the identity and use other trig identities, to convert that side into the other side therefore showing that they are equal.

Guidelines (make sure you have your identities handy)

1.) Start with the most complicated side.
2.) Look for helpful identities that will simplify your side.
3.) Look for algebraic operations that might help (adding, distributing, factoring, rationalizing)
4.) If you can think of nothing else, try changing everything to sines and cosines and see if that makes a difference.
5.) Always keep an eye on the other side of the equation to ensure that you are getting closer.

Example 1: Prove the following trigonometric identity.

$$\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} = 1 - \tan^2\theta$$

Proof:

-Before starting, please always make sure that you are justifying each step of you are taking., this means writing down the rule that you are employing at each step.

-We will start with the left side since it looks more complicated.

$$\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta}$$

-We can see that since the numerator is a difference of squares, it can be **factored**.

$$\frac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$$

-The left factor in the numerator can be simplified using a **Pythagorean identity**.

$$\frac{(1)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta} \quad \rightarrow \quad \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

-Now, the best thing that we can do is **split the fractions**.

$$\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

-This first term **simplifies to 1.**

$$1 - \frac{\sin^2\theta}{\cos^2\theta}$$

-Lastly, the right term will simplify using a **ratio identity**.

$$1 - \tan^2\theta$$

-This concludes our proof as we have converted the right side into the left. ///

Example 2: Prove the following trigonometric identities.

$$1 + \cos\varphi = \frac{\sin^2\varphi}{1 - \cos\varphi}$$

Proof:

-We will start on the right side since it is more complicated.

$$\frac{\sin^2\varphi}{1 - \cos\varphi}$$

-We first try altering the numerator using the **Pythagorean identity**.

$$\frac{1 - \cos^2\varphi}{1 - \cos\varphi}$$

-Using a difference of squares, we can **factor** the numerator.

$$\frac{(1 - \cos\varphi)(1 + \cos\varphi)}{1 - \cos\varphi}$$

-We can see that **the left term in the numerator will cancel** with the denominator to leave us the below term.

$$1 + \cos\varphi$$

-This completes our proof. ///

Example 3: Prove the following trigonometric identity.

$$\frac{\sin\alpha}{1+\cos\alpha}+\frac{1+\cos\alpha}{\sin\alpha}=2\csc\alpha$$

Proof:

-We start on the left side of the equation.

$$\frac{\sin\alpha}{1+\cos\alpha}+\frac{1+\cos\alpha}{\sin\alpha}$$

-We will combine these fractions by getting **a common denominator**.

$$\left(\frac{\sin\alpha}{\sin\alpha}\right)\frac{\sin\alpha}{1+\cos\alpha}+\left(\frac{1+\cos\alpha}{1+\cos\alpha}\right)\frac{1+\cos\alpha}{\sin\alpha}$$

-Next we **distribute** everything to get the following expression.

$$\frac{\sin^2\alpha}{\sin\alpha\,(1+\cos\alpha)}+\frac{1+2\cos\alpha+\cos^2\alpha}{\sin\alpha\,(1+\cos\alpha)}$$

-**Add the fractions** and combine the numerators.

$$\frac{1+2\cos\alpha+\cos^2\alpha+\sin^2\alpha}{\sin\alpha\,(1+\cos\alpha)}$$

-The last two terms in the numerator can be combined using a **Pythagorean identity**.

$$\frac{1+2\cos\alpha+1}{\sin\alpha\,(1+\cos\alpha)}$$

-We **combine the like terms** in the numerator.

$$\frac{2+2\cos\alpha}{\sin\alpha\,(1+\cos\alpha)}$$

-Next we will **factor the 2** out of the numerator.

$$\frac{2(1+\cos\alpha)}{\sin\alpha\,(1+\cos\alpha)}$$

-Now we can **cancel $1+\cos\alpha$ in the numerator and denominator** giving us to following.

$$\frac{2}{\sin\alpha}$$

-Lastly, we employ the **reciprocal identity**.

$$2\csc\alpha$$

-The proof is complete. ///

Example 4: Prove the following trigonometric identity.

$$\frac{1+\sin t}{\cos t}=\frac{\cos t}{1-\sin t}$$

Proof:

-In this case, both sides look to be of equal difficulty so, it doesn't really matter which one we choose.

-We arbitrarily choose the left side.

$$\frac{1+\sin t}{\cos t}$$

-While initially there don't seem to be any good options, we are going to employ a useful trick that you may need to use in the future so pay close attention.

-We are going to **rationalize the numerator**.

$$\frac{1+\sin t}{\cos t}\left(\frac{1-\sin t}{1-\sin t}\right)$$

-We **distribute** the numerator.

$$\frac{1-\sin^2 t}{\cos t\,(1-\sin t)}$$

-We can nor apply a **Pythagorean identity** to the numerator.

$$\frac{\cos^2 t}{\cos t\,(1-\sin t)}$$

-Next, the cosine in the denominator will **cancel** with one of the cosines in the numerator.

$$\frac{\cos t}{1-\sin t}$$

-The proof is complete. ///

Example 5: Prove the following trigonometric identity.

$$\sec\theta+\tan\theta=\frac{1}{\sec\theta-\tan\theta}$$

Proof:

We will start with the right side of the identity.

$$\frac{1}{\sec\theta-\tan\theta}$$

-There are multiple choices for our first step but let's go ahead and **rationalize** the denominator.

$$\left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right)\frac{1}{\sec\theta - \tan\theta}$$

-We now **distribute.**

$$\frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta}$$

-According to a **Pythagorean identity**, the denominator reduces to one.

$$\frac{\sec\theta + \tan\theta}{1} \quad \rightarrow \quad \sec\theta + \tan\theta$$

-The proof is complete. ///

Example 6: Prove the following trigonometric identity.

$$\tan\beta + \cot\beta = \sec\beta\csc\beta$$

Proof:

-We choose the left side of the identity to start with.

$$\tan\beta + \cot\beta$$

-This a problem where the best thing to do is to convert everything to sines and cosines using the **ratio identities.**

$$\frac{\sin\beta}{\cos\beta} + \frac{\cos\beta}{\sin\beta}$$

-Next, we want to **combine these two fractions by finding common denominators.**

$$\left(\frac{\sin\beta}{\sin\beta}\right)\frac{\sin\beta}{\cos\beta} + \left(\frac{\cos\beta}{\cos\beta}\right)\frac{\cos\beta}{\sin\beta}$$

-Once we **distribute,** we have the following expression.

$$\frac{\sin^2\beta + \cos^2\beta}{\sin\beta\cos\beta}$$

-We apply a **Pythagorean identity** in the numerator.

$$\frac{1}{\sin\beta\cos\beta}$$

-We now apply the **reciprocal identities** to get $\csc\beta\sec\beta \rightarrow \sec\beta\csc\beta$

-The proof is complete. ///

Example 7: Prove the following trigonometric identity.

$$\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \cos^3 x \sqrt{\sin x}$$

Proof:

We start with the left side.

$$\sin^{1/2} x \cos x - \sin^{5/2} x \cos x$$

The first thing that we can do is **factor out $\sin^{1/2} x$** from both terms.

$$\sin^{1/2} x \left(\cos x - \sin^{4/2} x \cos x\right) \quad \rightarrow \quad \sin^{1/2} x \,(\cos x - \sin^2 x \cos x)$$

Now, we can see that we can also **factor $\cos x$** out of the term.

$$\sin^{1/2} x \cos x \,(1 - \sin^2 x)$$

If we look at what is inside the parentheses, we can see a **Pythagorean identity** and make the following change.

$$\sin^{1/2} x \cos x \,(\cos^2 x)$$

If we **distribute the cos x** back into the parentheses, we get.

$$\sin^{1/2} x \cos^3 x$$

Since we know that a square root is the same as the ½ power, we can **rewrite** it below as follows.

$$\cos^3 x \sqrt{\sin x}$$

The proof is complete. ///