Solving Exponential Equations

-Exponential equation can be very fun to solve and they are simply equations that contain terms like b^x.

-The are 3 main methods that we will use to solve exponential equations and they are: related bases, substitution, and logarithms.

-Below we will look at some examples of each of these methods.

Relating Bases

-The methods of relating bases can be useful in solving certain types of equations and they are based on the simple principal that is if $b^x = b^y$, then $x = y$.

-The only trick is that sometimes, you have to manipulate one or both bases in order to make them the same.

Example 1: Solve the following exponential equation by relating bases.

$$7^{\frac{3x-2}{4}} = \sqrt{7}$$

SOLUTION:

-First, we remember that a square root is really just a ½ power, so we can rewrite our equation below.

$$7^{\frac{3x-2}{4}} = 7^{\frac{1}{2}}$$

-Now that they have the same base, we can simply set their exponents equal to each other and solve.

$$\frac{3x-2}{4} = \frac{1}{2}$$

-We have a simple algebraic equation and we multiply both side by 4 to get.

$$3x - 2 = 2$$

-We add 2 to both sides.

$$3x = 4$$

-Lastly, we divide by 3 to obtain the following.

$$x = \frac{4}{3}$$

Example 2: Solve the following exponential equation by relating bases.

$$32^{x-2} = 16^{x+1}$$

SOLUTION:

-This first thing that we may notice, is that 32 is not a power of 16 and 16 is not a power of 32, so how can we make the bases the same?

-Well, it turns out that they are both powers of 2, $2^5 = 32$ and $2^4 = 16$.

-So, we are going to rewrite our equation as follows.

$$(2^5)^{x-2} = (2^4)^{x+1}$$

-We know that when we have a power to a power, we multiply, so we are going to distribute the 5 and the 4.

$$2^{5x-10} = 2^{4x+4}$$

-Now that the bases are equal, we can set the exponents equal to one another and solve.

$$5x - 10 = 4x + 4$$

-Add 10 to both sides

$$5x = 4x + 14$$

-Subtract 4x from both sides and we get our solution.

$$x = 14$$

Using Logarithms to Solve Exponential Equations.

- Logarithms are a powerful problem-solving tool and can be used to solve exponential equations in situations when bases cannot be related.
- In this method you simply use an appropriate logarithm to undo the exponent and isolate x, or you use the properties of logarithms to pull x down and solve for it.
- Below we will look at examples of each.

Example 1: Solve the following exponential equation, $5^x = 12$

SOLUTION:

-This is a very simple problem, we cannot relate the bases, so we will use a log base 5 on both sides to undo the exponent and isolate x.

$$\log_5(5^x) = \log_5(12)$$

-The log base 5 and the 5 cancels out leaving us with the following.

$$x = \log_5(12)$$

-Using the calculator, you could find a decimal approximation for this, but since this an exact answer, in this tutorial, we will leave the solution as $x = \log_5(12)$

Example 2: Solve the following exponential equation, round your answer to 3 decimal places.

$$2^{x-1} = 3^{x+2}$$

SOLUTION:

-Since the bases cannot be related, we are going to use the power rule to pull the exponents down and solve for x.

-Since the is no one logarithm that will undo both 2 and 3, we will simply use a natural logarithm in this case.

-First, we apply a natural log on both sides.

$$\ln(2^{x-1}) = \ln(3^{x+2})$$

-Now, we can use the power rule to pull the powers out front.

$$(x-1)\ln(2) = (x+2)\ln(3)$$

-We distribute the natural logs.

$$x\ln(2) - \ln(2) = x\ln(3) + 2\ln(3)$$

-We need our x terms on the same side, so, we subtract $x\ln(3)$ to the left side.

$$x\ln(2) - \ln(2) - x\ln(3) = 2\ln(3)$$

-We now add $\ln(2)$ to both sides.

$$x\ln(2) - x\ln(3) = 2\ln(3) + \ln(2)$$

-We factor an x from both terms on the left.

$$x[\ln(2) - \ln(3)] = 2\ln(3) + \ln(2)$$

-Lastly, we divide both sides by $[\ln(2) - \ln(3)]$, getting the following.

$$x = \frac{2\ln(3) + \ln(2)}{\ln(2) - \ln(3)}$$

-If we punch this in the calculator and round, we get $x \approx -7.129$.

Substitution

-This next method is one you have probably seen before.

-There are certain equations that that are factorable, but they don't like to be factored.

-Substitution is method where we substitute u for the equation variable and trick it into being easily factored.

Example 1: Solve the following exponential equations using substitution.

$$2^{2x} - 4 \cdot 2^x - 21 = 0$$

SOLUTION:

-The first thing that we will do is rewrite to make our substitution easier.

$$(2^x)^2 - 4(2^x) - 21 = 0$$

-Next, we can say that $u = 2^x$ and substitute u into the equation giving us.

$$u^2 - 4u - 21 = 0$$

-Now we can see that we have a simply algebraic equation that will factor below.

$$(u - 7)(u + 3) = 0$$

-From this we get the solutions that $u = 7$ and $u = -3$.

-Our solution cannot be in terms of u, but we remember that $u = 2^x$. Therefore, we substitute the values that we got for u into this equation getting the following two equations.

$$7 = 2^x \quad \& \quad -3 = 2^x$$

-Let's look at $-3 = 2^x$, to solve this equation, we will need to use a log base to in order to isolate x, so let apply a log base 2 on both sides.

$$\log_2(-3) = \log_2(2^x)$$

-At this point you should be seeing a contradiction, you CANNOT put a negative inside a logarithm, therefore we cannot solve this, so we throw -3 away and focus on the other potential solution.

-The only other option for us, is to solve the equation, $7 = 2^x$, we will do this similarly using a log base 2 on booth sides of the equation.

$$\log_2(7) = \log_2(2^x)$$

-As expected, the log base 2 and the 2 cancels out leaving us with the following equation.

$$\log_2 7 = x$$

-Therefore, the exact solution to our equation is $x = \log_2 7$.

Example 2: Solve the following exponential equation using substitution.

$$e^{4x} + 5e^{2x} - 24 = 0$$

SOLUTION:

-First, we begin by rewriting our equation as follows.

$$(e^{2x})^2 + 5(e^{2x}) - 24 = 0$$

-Next, we say that $u = e^{2x}$ and make our substitution below.

$$u^2 + 5u - 24 = 0$$

-This is now a simple algebraic equation that can be factored below.

$$(u + 8)(u - 3) = 0$$

-From this we get the following solutions.

$$u = -8, 3$$

-Since we need out solution in terms of x, we remember that $u = e^{2x}$, we substitute both values of u into this equation to get x.

$$3 = e^{2x} \quad \& \quad -8 = e^{2x}$$

-We can throw the second equation away since it is a contradiction and now, we can solve the firt equation using a natural logarithm.

$$\ln(3) = \ln(e^{2x})$$

-The ln and the cancel giving us the following equation.

$$\ln(3) = 2x$$

-Lastly, we divide by 2 and get the following solution.

$$x = \frac{1}{2}\ln(3)$$

Example 3: Solve the following exponential equation using substitution.

$$3^{3x} - 4 \cdot 3^{2x} + 2 \cdot 3^{x} = 8$$

SOLUTION:

-The first step is always to rewrite our equation in such a way that makes our substitution obvious.

$$(3^x)^3 - 4 \cdot (3^x)^2 + 2 \cdot 3^x = 8$$

-We can see that we will make the substitution $u = 3^x$ to give us the following polynomial.

$$u^3 - 4u^2 + 2u = 8$$

-Since this is a polynomial equation, we need to set it equal to 0.

$$u^3 - 4u^2 + 2u - 8 = 0$$

-We will factor this by grouping as seen below.

$$(u^3 - 4u^2) + (2u - 8) = 0$$

$$u^2(u - 4) + 2(u - 4) = 0$$

$$(u - 4)(u^2 + 2) = 0$$

$$u = 4 \quad u^2 = -2$$

$$u = 4 \quad u = \pm i\sqrt{2}$$

-We can throw away the imaginary solutions, so this leaves a solution of $u = 4$.

-Since x is our solution, not u, and $u = 3^x$, we back substitute into the following.

$$4 = 3^x$$

-We solve this using a log base 3 on both sides.

$$\log_3(4) = \log_3(3^x)$$

-The log base 3 and the 3 cancels out giving us.

$$\log_3(4) = x$$

-You may punch this into your calculator for a decimal solution, but we will leave $x = \log_3(4)$ as the exact solution.