# Solving Logarithmic Equations

-A logarithmic equation is simply an equation that contains a logarithm.

-When solving a logarithmic equation, you want to make sure that you contract any logs on either side of the equation.

-Once all logs are contracted, exponentiate to get rid of logs.

-Solve the resulting algebraic equation.

-Check for extraneous solutions.

**Example 1**: Solve the following logarithmic equation.

$$\log(x^2 - x - 5) = 0$$

SOLUTION:

-Since all logs are contracted, we will start by exponentiating. Since this is a log base 10, we will raise 10 to the power of both sides of this equation.

$$10^{\log(x^2 - x - 5)} = 10^0$$

-On the left side of the equation the 10 and the log base 10 will cancel out, on the right side $10^0 = 1$. Therefore, this will simplify to the equation seen below.

$$x^2 - x - 5 = 1$$

-Now that we have a quadratic, we will set it equal to 0.

$$x^2 - x - 6 = 0$$

-Next, we factor and solve the quadratic.

$$(x - 3)(x + 2) = 0$$

$$x = 3 \qquad x = -2$$

-Next, we check both solutions by plugging them into the original log equation.

| | | | | |
|---|---|---|---|---|
| $x = 3$ | $\log((3)^2 - (3) - 5) = 0$ | $\log(1) = 0$ | $0 = 0$ | ☑ |
| $x = -2$ | $\log((-2)^2 - (-2) - 5) = 0$ | $\log(1) = 0$ | $0 = 0$ | ☑ |

-Since both solutions check out, our solutions are 3, and -2.

**Example 2**: Solve the following logarithmic equation.

$$\ln(2x-3)-\ln(x+5)=0$$

SOLUTION:

-We begin by contracting the logarithms on the left side using the quotient rule.

$$\ln\left(\frac{2x-3}{x+5}\right)=0$$

-Next, we exponentiate on the sides by raising e to the power of each side of this equation.

$$e^{\ln\left(\frac{2x-3}{x+5}\right)}=e^{0}$$

-The e and the ln cancel out and $e^{0}=1$ so, our equation simplifies to the equation below.

$$\frac{2x-3}{x+5}=1$$

-Next, we will multiply both sides by (x+5) in order to get rid of the fraction.

$$(x+5)\left(\frac{2x-3}{x+5}\right)=(x+5)(1)$$

-This simplifies as the following equation.

$$2x-3=x+5$$

-To solve this, we add 3 to both sides of the equation.

$$2x=x+8$$

-Then, we subtract x from both sides.

$$x=8$$

-Next, we check this solution by plugging into the original logarithm.

| | | | | |
|---|---|---|---|---|
| $x=8$ | $\ln(2(8)-3)-\ln((8)+5)=0$ | $\ln 13-\ln 13=0$ | $0=0$ | ☑ |

-Since this check's out, the solution to this equation is $x=8$.

**Example 3**: Solve the following logarithmic equation.

$$\log_6(x+2) + \log_6(x-3) = 1$$

SOLUTION:

-The first thing is to contract the logarithms on the left side by reversing the product rule.

$$\log_6[(x+2)(x-3)] = 1$$

-Next, we will exponentiate by raise 6 to the power of both sides of this equation thereby undoing the logarithm.

$$6^{\log_6[(x+2)(x-3)]} = 6^1$$

-This will simplify as the below equation.

$$(x+2)(x-3) = 6$$

-Now, we will FOIL and simplify the left side of the equation.

$$x^2 - 3x + 2x - 6 = 6 \quad \rightarrow \quad x^2 - x - 6 = 6$$

-We need to set it equal to 0, so, we will subtract 6 from both sides to get.

$$x^2 - x - 12 = 0$$

-Next, we factor and solve.

$$(x-4)(x+3) = 0$$

$$x = 4 \quad , \quad x = -3$$

-We will test these values in the table below.

| | | | | |
|---|---|---|---|---|
| $x = 4$ | $\log_6(4+2) + \log_6(4-3) = 1$ | $\log_6 6 + \log_6 1 = 1$ | $1 = 1$ | ☑ |
| $x = -3$ | $\log_6(-3+2) + \log_6(-3-3) = 1$ | $\log_6 -1 + \log_6 -6 = 1$ | $\emptyset$ | ☒ |

-3 will not compute because it creates a negative in the logarithm, it is therefore not a solution.

-Therefore, the solution to our logarithmic equation is $x = 4$.

**Example 4:** Solve the following logarithmic equation.

$$\log_7 3x + \log_7(2x - 1) = \log_7(16x - 10)$$

SOLUTION:

-First, we contract the left side of the logarithm by reversing the product rule.

$$\log_7[3x(2x - 1)] = \log_7(16x - 10)$$

-Now, we exponentiate on both sides using 7.

$$7^{\log_7[3x(2x-1)]} = 7^{\log_7(16x-10)}$$

-Both logs cancel leaving us with the following equation.

$$3x(2x - 1) = 16x - 10$$

-Below, we will solve this equation using methods from algebra.

$$3x(2x - 1) = 16x - 10$$

$$6x^2 - 3x = 16x - 10$$

$$6x^2 - 19x + 10 = 0$$

$$(6x^2 - 4x) + (-15x + 10) = 0$$

$$2x(3x - 2) - 5(3x - 2) = 0$$

$$(3x - 2)(2x - 5) = 0$$

$$x = \frac{2}{3} \quad , \quad x = \frac{5}{2}$$

-Next, we test both solutions.

| | | | | |
|---|---|---|---|---|
| $x = \frac{2}{3}$ | $\log_7 3\left(\frac{2}{3}\right) + \log_7(2\left(\frac{2}{3}\right) - 1) = \log_7(16\left(\frac{2}{3}\right) - 10)$ | $\log_7\left(\frac{2}{3}\right) = \log_7\left(\frac{2}{3}\right)$ | $\frac{2}{3} = \frac{2}{3}$ | ☑ |
| $x = \frac{5}{2}$ | $\log_7 3\left(\frac{5}{2}\right) + \log_7(2\left(\frac{5}{2}\right) - 1) = \log_7(16\left(\frac{5}{2}\right) - 10)$ | $\log_7 30 = \log_7 30$ | $30 = 30$ | ☑ |

-Since both solutions check out, our solutions are $x = \frac{2}{3}$ and $x = \frac{5}{2}$.