Solving Polynomial and Rational Inequalities

Polynomial Inequalities

-You are probably used to dealing with linear inequalities like, $2x - 3 \geq 7(x + 1) - 3$.

-As you know these problems are relatively simple and follow a lot of the basic rules of algebra.

-Solving a linear inequality is a lot like solving a linear equation.

-But how do you solve an inequality like this, $x^2 + 2x + 1 \leq 0$?

-This is a polynomial inequality and the process of solving it is actually quite a bit different from solving a polynomial equation.

Steps for solving Polynomial Inequalities.

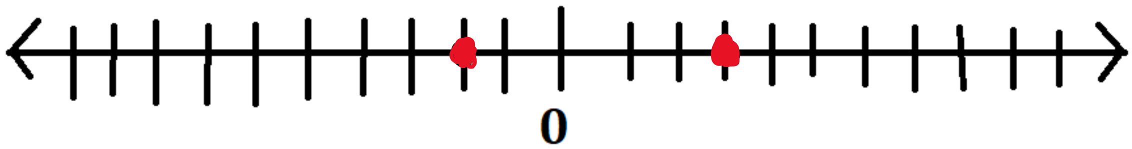
1.) Make sure that 0 is isolated on one side, if it already is, then start with step 2.
2.) Determine whether you are looking for values that make your polynomial negative or positive.
3.) If the polynomial is not factored, factor it to find all solutions (these solutions will be called **Boundary Points**)
4.) Place your boundary points on the number and remember to use closed circles for $\leq$ $\geq$ and open circles for $<>$.
5.) Your boundary points have cut your number line into intervals, write each interval separately.
6.) Choose a **Test Point** from each interval. (a test point can literally be any number in that interval)
7.) Set up a table and plug each test point into the polynomial inequality to see if it is satisfied.
8.) If a point works, it means that the interval it comes from is a solution.
9.) Unite all the solution intervals and you have your solution in interval notation.

Example 1: Solve the following polynomial inequality, $(x - 3)(x + 2) \geq 0$.

SOLUTION:

1.) This step is already complete, so we move to step 2.
2.) Because this polynomial is greater than or equal to 0, we conclude that we are looking for **POSITIVE** values.
3.) Since the polynomial is already factored, we can see that the solution set is $\{3, -2\}$. Therefore, 3 and -2 are our boundary points.

4.) We will now place these two points on the number line using closed circles because this is not a strict inequality.



5.) We can now see that our number line is cut into the following 3 intervals.

$$(-\infty, -2], [-2, 3], and\ [3, \infty)$$

6.) Now we will choose our test points. From $(-\infty, -2]$ we choose -3, from $[-2, 3]$ we choose 0, and from $[3, \infty)$ we choose 4.

7.) Our test points are -3, 0, and 4. We will now test each point by plugging it into the polynomial. Remember that we are looking for **POSITIVE** values only, if we get a negative value, then that interval is *not* part of our solution. We will see the results in the following table.

Interval	Test Point	$(x-3)(x+2)$	Sign
$(-\infty, -2]$	-3	$(-3-2)(-3+2) = 6$	$+$
$[-2, 3]$	0	$(0-3)(0+2) = -6$	$-$
$[3, \infty)$	4	$(4-3)(4+2) = 6$	$+$

8.) Since we were looking for solutions with positives, we choose the positive intervals and unite then to for a solution of $(-\infty, -2] \cup [3, \infty)$.

Example 2: Solve the following polynomial inequality, $x^2 - 25 < 0$.

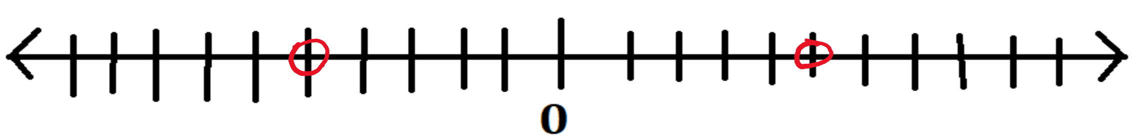
Solution:

1.) This step is already complete, so we move to step 2.

2.) Since the polynomial must be less than 0, we conclude that we are looking for negatives.

3.) We will factor this using the difference of squares to get $(x+5)(x-5)$, we can see that the solution set is $\{-5, 5\}$. This means that -5 and 5 are the boundary points.

4.) We will now place these points on the number line using open circles because these inequalities are strict.



5.) We can see that our number line is cut into the following 3 intervals.

$$(-\infty, -5), (-5, 5), (5, \infty)$$

6.) Now we choose our test points, from $(-\infty, -5)$ we choose -6, from $(-5, 5)$ we choose 0, and from $(5, \infty)$ we choose 6.

7.) Our test points are -6, 0, and 6. We will now test each point by plugging it into the polynomial. Remember that we are looking for **NEGATIVE** values only, if we get a positive value, then that interval is ***not*** part of our solution. We will see the results in the following table.

Interval	Test Point	$x^2 - 25$	Sign
$(-\infty, -\mathbf{5})$	-6	$(-6)^2 - 25 = 11$	$+$
$(-\mathbf{5}, \mathbf{5})$	0	$(0)^2 - 25 = -25$	$-$
$(\mathbf{5}, \infty)$	6	$(6)^2 - 25 = 11$	$+$

8.) Since we are looking for the negative values, we choose the interval that is negative, the solution is $(-5, 5)$.

Example 3: Solve the following polynomial inequality, $x^3 \geq x$.

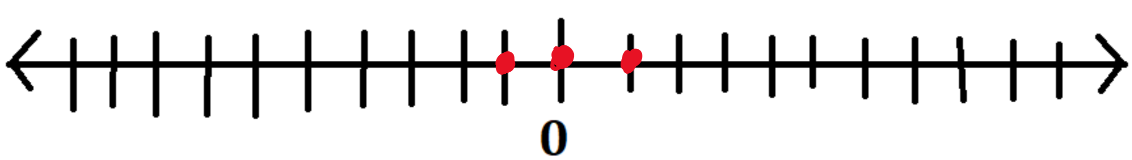
SOLUTION:

1.) In order to isolate 0 on one side of the inequality, we will subtract x over to the other side making our inequality, $x^3 - x \geq 0$.

2.) From our previous step, we can see that our polynomial must now be greater than 0, therefore we conclude that we are looking for positives.

3.) To factor $x^3 - x$, we first notice that both terms share an x, therefore we will first pull an x out getting, $x(x^2 - 1)$. Now, we can see that what is in the parentheses is a perfect square and can be factored to obtain, $x(x + 1)(x - 1)$. Now that our polynomial is completely factored, we can see that if we set each term equal to 0, we will obtain the following solution set, $\{-1, 0, 1\}$. This means that -1, 0, and 1 are our boundary points.

4.) We will now place these points on the number line using closed circles because the inequality is not strict.



5.) We can see that our number line is cut into the following 4 intervals.

$$(-\infty, -1], [-1, 0], [0, 1], [1, \infty)$$

6.) We can now choose a test point from each interval. From $(-\infty, -1]$, we choose -2, from $[-1, 0]$, we choose -0.5, from $[0, 1]$, we choose 0.5, and from $[1, \infty)$, we choose 2.
7.) Our test points are -2, -0.5, 0.5, and 2. We will test these points by plugging them into the function in the table below. Before doing this, please remember that we are looking for values that make our polynomial **POSITIVE**, if it makes our polynomial negative, it is not a solution.

Interval	Test Point	$x^3 - x$	Sign
$(-\infty, -\mathbf{1}]$	-2	$(-2)^3 - (-2) = -6$	$-$
$[-\mathbf{1}, \mathbf{0}]$	-0.5	$(-0.5)^3 - (-0.5) = 0.375$	$+$
$[\mathbf{0}, \mathbf{1}]$	0.5	$(0.5)^3 - (0.5) = -0.375$	$-$
$[\mathbf{1}, \infty)$	2	$(2)^3 - (2) = 6$	$+$

8.) Since we are looking for positive intervals, we unite the positive intervals to get a solution of $[-1, 0] \cup [1, \infty)$.

Rational Inequalities

- A rational inequality is an inequality that involves fractions, below are some examples of rational inequalities.

$$\frac{x-2}{x+1} > 0 \qquad \frac{x^2-3}{x+7} \leq \frac{x-4}{x+5}$$

- While these may look quite a bit different from polynomial inequalities, the process of solving them is actually quite similar.

Steps for solving Rational Inequalities

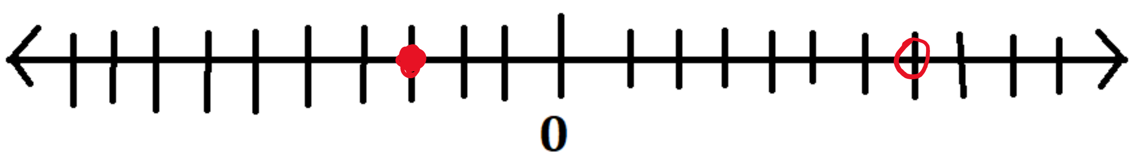
1.) Make sure that 0 is isolated on one side, if it already is, then start with step 2.
2.) Make sure that all terms are consolidated into one fraction.
3.) Determine whether you are looking for values that make your Rational Expression negative or positive.
4.) Set the numerator and denominator equal to 0 and solve to get **Boundary Points** (you may have to factor one or both).
5.) Place your boundary points on the number line and use open or closed circles depending on if the inequality is strict or not. That being said, because you cannot actually have 0 in the denominator, **all boundary points coming from the denominator will have open circles** regardless of the inequality.
6.) Your boundary points have cut your number line into intervals, write each interval separately.
7.) Choose a **Test Point** from each interval. (a test point can literally be any number in that interval)
8.) Set up a table and plug each test point into the rational expression to see if it is satisfied.
9.) If a point works, it means that the interval it comes from is a solution. So, choose all intervals that work.

Example 1: Solve the following rational inequality.

$$\frac{x+3}{x-7} \leq 0$$

SOLUTION:

1.) Since 0 is already isolated on one side we can move to step 2.
2.) All terms are consolidated into one fraction, so we can move to step 3.
3.) Since the rational expression is less than or equal to 0, we conclude that we are looking for **negatives**.
4.) Now, we set the numerator and denominator equal to 0 and solve. This means that $x + 3 = 0$ and $x - 7 = 0$. If we solve both of these, we get that -3 and 7 are boundary points.
5.) Now we will place these on the number line remembering that because this is not a strict inequality, -3 will have a closed circle, but 7 will have an open circle because it comes from the denominator.



6.) Our number line is now cut into the 3 intervals that we see below.

$$(-\infty, -3], [-3, 7), (7, \infty)$$

7.) Now, we will pick a test point from each interval. From $(-\infty, -3]$ we choose -4, from $[-3, 7)$ we choose 0, and from $(7, \infty)$ we choose 8.
8.) Our test points are -4, 0, and 8, now, we will plug them into our rational expression and keep in mind, we are looking for NEGATIVES.

Interval	**Test Point**	$\boldsymbol{\frac{x+3}{x-7}}$	**Sign**
$(-\infty, -\mathbf{3}]$	-4	$\frac{(-4)+3}{(-4)-7} = \frac{-1}{-11} = \frac{1}{11}$	$+$
$[-\mathbf{3}, \mathbf{7})$	0	$\frac{(0)+3}{(0)-7} = \frac{3}{-7} = -\frac{3}{7}$	$-$
$(\mathbf{7}, \infty)$	8	$\frac{(8)+3}{(8)-7} = \frac{11}{1} = 11$	$+$

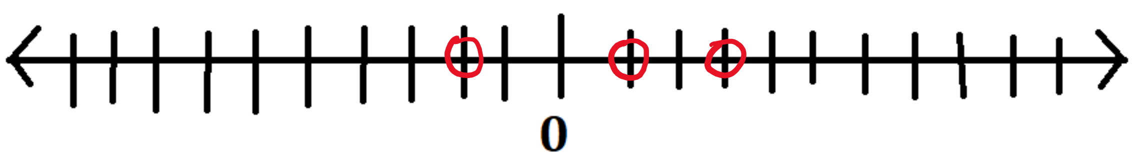
9.) Since we were looking for negatives, we can see that there is only one interval that provides a negative, therefore, the solution is $[-3, 7)$.

Example 2: Solve the following rational inequality.

$$\frac{x^2+x-2}{x-3}>0$$

SOLUTION:

1.) Since 0 is already isolated on one side of the inequality, we can move to step 2.
2.) Since the terms are already consolidated into one fraction, we can move to step 3.
3.) Since the rational expression is greater than 0, we conclude that we are looking for **positives**.
4.) We now set both the numerator and denominator equal to 0 and solve to get the boundary points. For the denominator, $x-3=0$, gives a solution of 3. For the numerator, $x^2+x-2=0$, this can be factored like, $(x-1)(x+2)=0$, this gives boundary points of 1, and -2.
5.) Our boundary points are -2, 1, and 3. We will now place these on the number line using open circles for all points.



6.) We can see that our line is now cut into 4 intervals as shown below.

$$(-\infty,-2)\,,(-2,1)\ \,,(1,3)\,,(3,\infty)$$

7.) We will now choose a test point from each interval. From $(-\infty,-2)$, we choose -3, from $(-2,1)$, we choose 0, from $(1,3)$, we choose 2, and from $(3,\infty)$, we choose 4.
8.) Our test points are -3, 0, 2, and 4. We will check these in the table below and keep in mind that we are looking for positives.

Interval	**Test point**	$\frac{x^2+x-2}{x-3}$	**Sign**
$(-\infty,-2)$	-3	$\frac{(-3)^2+(-3)-2}{(-3)-3}=\frac{4}{-6}=-\frac{2}{3}$	$-$
$(-2,1)$	0	$\frac{(0)^2+(0)-2}{(0)-3}=\frac{-2}{-3}=\frac{2}{3}$	$+$
$(1,3)$	2	$\frac{(2)^2+(2)-2}{(2)-3}=\frac{4}{-1}=-4$	$-$
$(3,\infty)$	4	$\frac{(4)^2+(4)-2}{(4)-3}=\frac{18}{1}=18$	$+$

9.) We unite the positive intervals to get a solution of $(-2,1)\cup(3,\infty)$.

Example 3: Solve the following rational inequality.

$$\frac{2}{x+1} \geq \frac{1}{x-2}$$

SOLUTION:

1.) We will first move one term to the other side to isolate 0.

$$\frac{2}{x+1} - \frac{1}{x-2} \geq 0$$

2.) We will now combine our terms by finding a common denominator, $(x+1)(x-2)$.

$$\left(\frac{x-2}{x-2}\right)\frac{2}{x+1} - \left(\frac{x+1}{x+1}\right)\frac{1}{x-2} = \frac{2(x-2) - 1(x+1)}{(x+1)(x-2)} = \frac{2x-4-x-1}{(x+1)(x-2)}$$

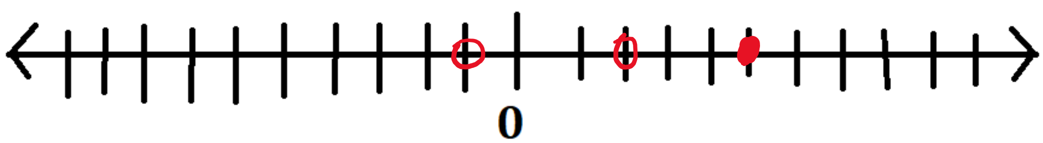
After further simplification, we can arrive at the following result.

$$\frac{x-5}{(x+1)(x-2)} \geq 0$$

3.) Since our rational expression must be greater than or equal to 0, we conclude that we are looking for **positives**.

4.) We now set our denominator and numerator equal to 0, to find boundary points. From the numerator we get 5 as a boundary point and from the denominator we get -1 and 2 as boundary points.

5.) We have -1, 2, and 5 as boundary points, we will now place them on the number line. Because the inequality is not strict, 5 will have a closed circle, but -1 and 2 will have open circles because they come from the denominator.



6.) We can see that our number line is cut into the following 4 pieces.

$$(-\infty, -1), \qquad (-1, 2), \qquad (2, 5], \qquad [5, \infty)$$

7.) Now we need to choose our test points. From $(-\infty, -1)$ we choose -2, from $(-1, 2)$, we choose 0, from $(2, 5]$, we choose 3, and lastly, from $[5, \infty)$, we choose 6.

8.) Our test points are -2, 0, 3, and 6. We will now test these points in the table shown below, please remember that we are looking for POSITIVE values in our expression.

Interval	Test Point	$\dfrac{\boldsymbol{x-5}}{\boldsymbol{(x+1)(x-2)}}$	Sign
$(-\infty,-1)$	-2	$\dfrac{(-2)-5}{((-2)+1)((-2)-2)}=\dfrac{-7}{4}=-\dfrac{7}{4}$	$-$
$(-1,2)$	0	$\dfrac{(0)-5}{((0)+1)((0)-2)}=\dfrac{-5}{-2}=\dfrac{5}{2}$	$+$
$(2,5]$	3	$\dfrac{(3)-5}{((3)+1)((3)-2)}=\dfrac{-2}{4}=-\dfrac{1}{2}$	$-$
$[5,\infty)$	6	$\dfrac{(6)-5}{((6)+1)((6)-2)}=\dfrac{1}{28}$	$+$

9.) Since we want the positives, we unite the positive intervals for a final solution of $(-1,2)\cup[5,\infty)$.

Common applications:

-You may be wondering why any of this is relevant and in what circumstance this skill would be useful.

-There is one application that stands out and it is one that may affect your life in the near future.

-The application is finding Domain for functions.

-In order to find the domain for the following functions, you would need polynomial or rational inequalities.

$$f(x)=\sqrt{x^2-3x+2} \qquad g(x)=\sqrt{\frac{x+1}{x-2}} \qquad f(x)=\frac{2x-3}{\sqrt{x^2-4}}$$