The Unit Circle

-The unit circle is a very useful tool that is used often by math, physics, and engineering students to calculate the values of various trigonometric functions.

-The equation for the unit circle is $x^2 + y^2 = 1$, it is a circle centered at the origin with a radius of 1.

-In this tutorial, we will review special right triangles and learn how to construct the unit circle.

Special Right Triangles

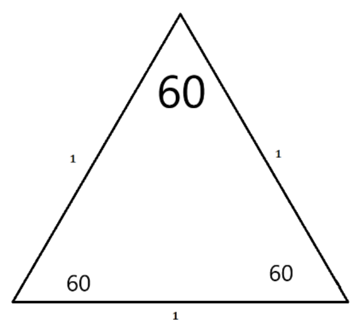
-We are going to examine the properties 45-45-90 and 30-60-90 triangles in this section.

-These are triangles that due to their interior angles follow certain patterns.

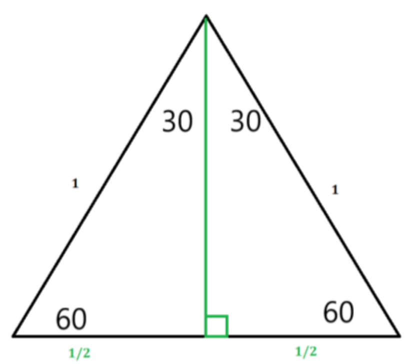
-Since the unit circle has a radius of 1, we will assume that all triangles have a hypotenuse of 1.

30-60-90

-We will start with an equilateral triangle that has side lengths of 1. Note that when a triangle is equilateral all its interior angles are 60 degrees.

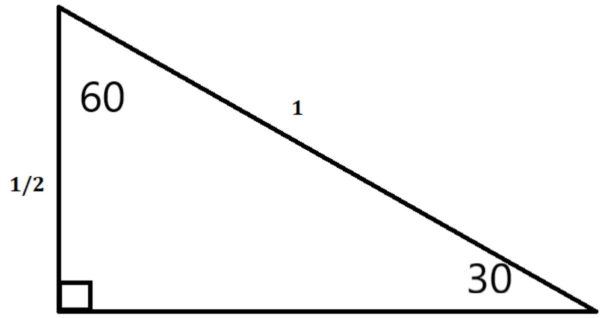


-Next, we cut this equilateral triangle right down the middle to create two right triangles.



-Notice that the bottom has been cut into two halves that each have a length of 1/2. Also, the top angle has been cut into two 30-degree angles.

-Next we take just one of these right triangles.



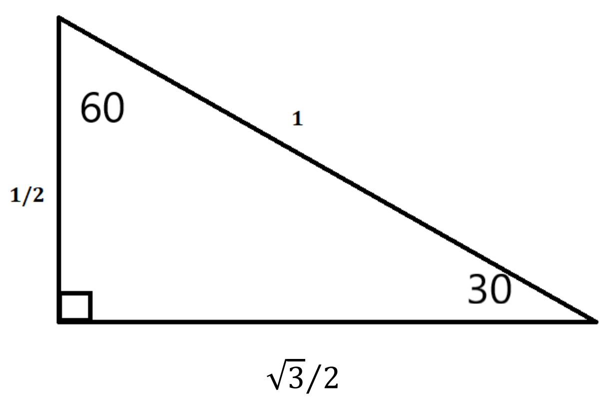
-We notice that the angles inside are 30-60-90, thus the term 30-60-90 triangle.

-We can also see that we are missing one side.

-Since this is a right triangle, we can employ Pythagorean theorem below.

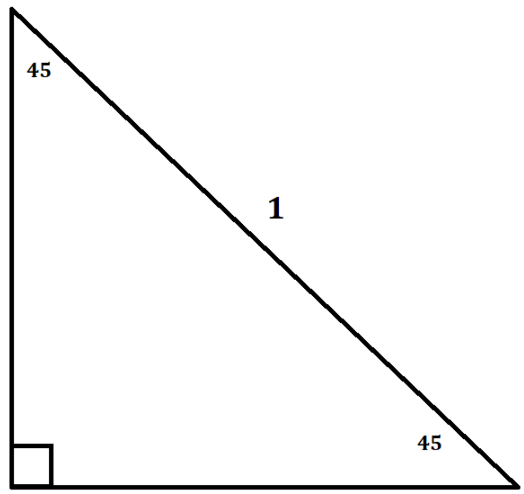
Apply Pythagorean Theorem.	$a^2 + \left(\frac{1}{2}\right)^2 = 1^2$
Simplify	$a^2 + \frac{1}{4} = 1$
Subtract ¼ from each side	$a^2 = \frac{3}{4}$
Take the square root of each side.	$a = \pm\frac{\sqrt{3}}{2}$
Choose the positive value.	$a = \frac{\sqrt{3}}{2}$

-This means that we now have the following triangle.



45-45-90

-These triangles much like the 30-60-90 triangle we looked at will have a hypotenuse of 1, but of course the interior angle will be 45-45-90.

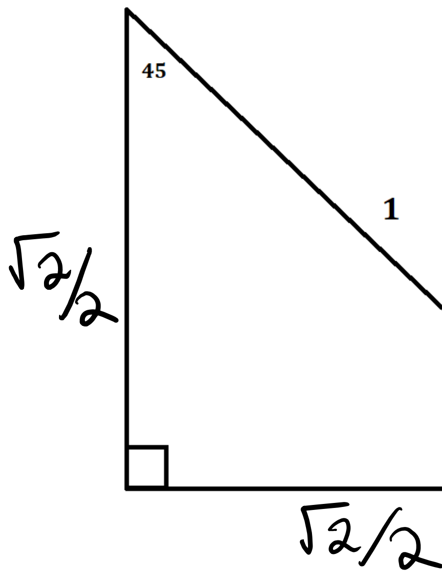


-If you dig up your memory from Geometry you will remember that since each leg of the triangle is opposite a 45-degree angle, then each leg must be the same length.

-Therefore, while we only know the hypotenuse, we can know that both legs are the same length, in fact, we will call them x, and solve for them using Pythagorean theorem.

Apply the Pythagorean Theorem	$x^2 + x^2 = 1^2$
Combine like terms	$2x^2 = 1$
Divide by 2.	$x^2 = \frac{1}{2}$
Take the square root of each side	$x = \pm\sqrt{\frac{1}{2}}$
Simplify	$x = \pm\frac{1}{\sqrt{2}}$
Choose the positive value	$x = \frac{1}{\sqrt{2}}$
Rationalize	$x = \frac{\sqrt{2}}{2}$

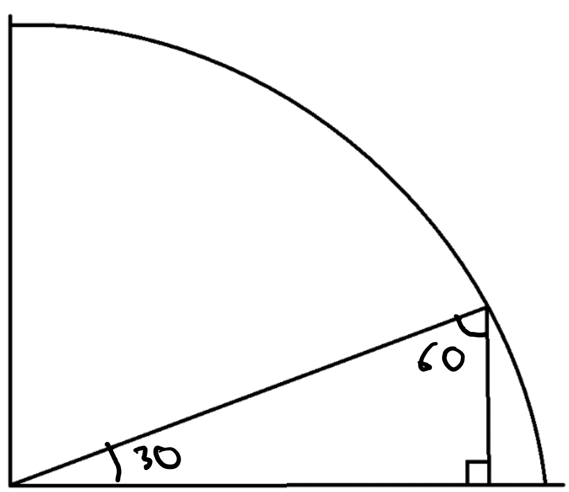
-So, we now know that a 45-45-90 triangle with a hypotenuse of 1 looks like the below triangle.



Constructing the 1st Quadrant of the Unit Circle

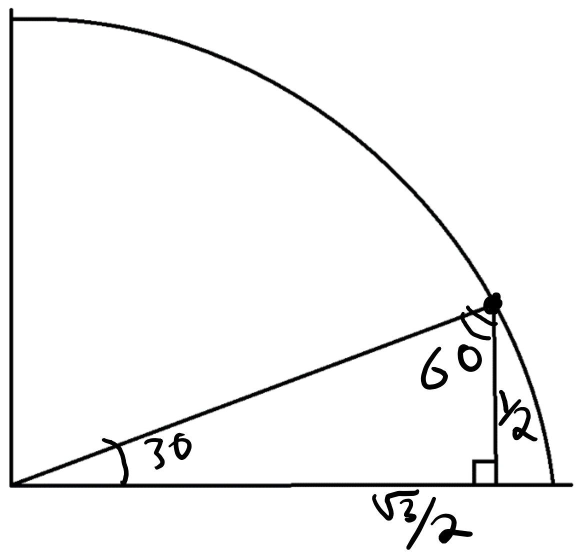
-Remember that the unit circle has a radius of 1

-The 1st angle that we place on the unit circle is going to be a 30-degree angle.

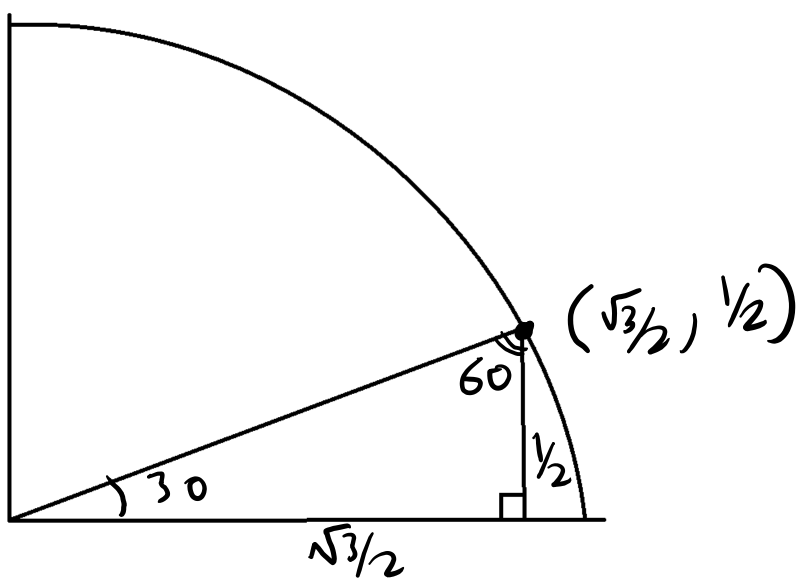


-We can see that that by adding the 30-degree angle, we are actually placing a 30-60-90-degree triangle in the unit circle.

-Since the hypotenuse is known to be 1 due to the circle's radius, then we also know the other two sides are as follows:



-We now need to write the ordered pair for the point on the triangle that touches the circle. Since the point has an x value of $\frac{\sqrt{3}}{2}$ and a y value of $\frac{1}{2}$, the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Hence the following sketch.



-Next, we are going to add a 45-degree angle, and keep in mind that it is really a 45-45-90 triangle that we are placing on the circle.

-Since you have seen how this works, hopefully the following step will make sense.

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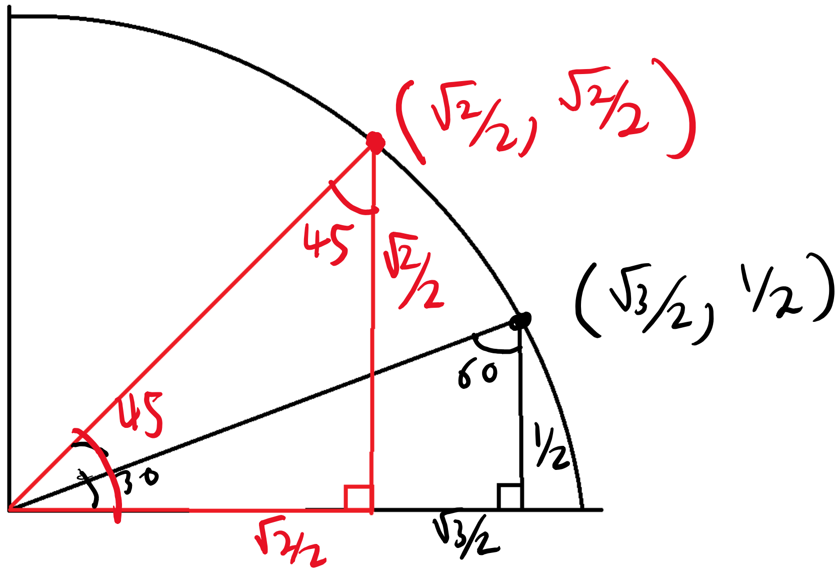
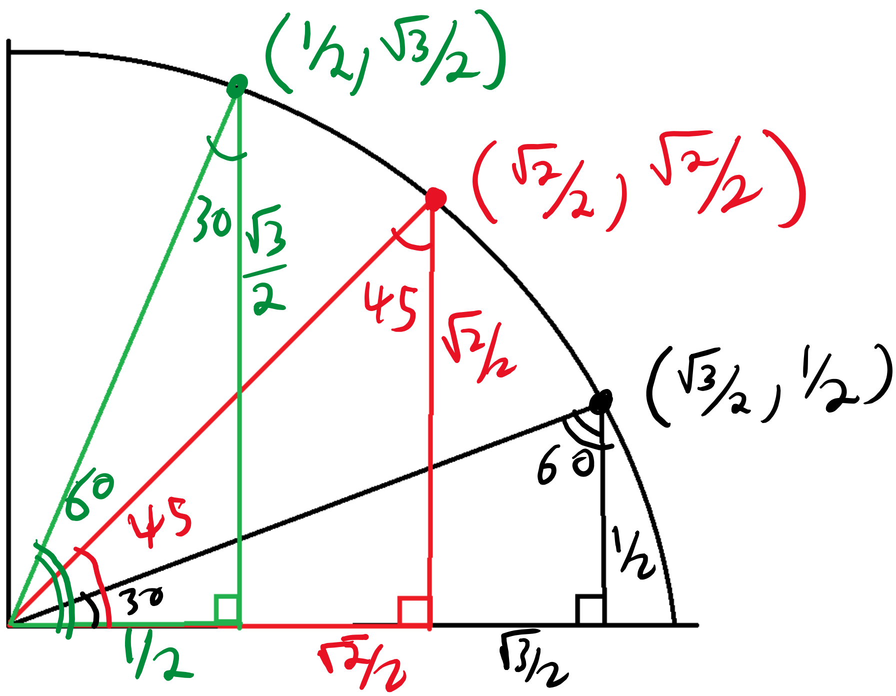


-You can see how we got the ordered pair $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in a similar fashion.

-Next, we add a 60-degree angle and as you may have guessed, this will be another 30-60-90 triangle with the sides flipped, it will result in the following.



-Now that we have constructed the 1st Quadrant of the unit circle, let's use it to calculate some values.

-Remember that because the radius of the circle is 1, the hypotenuse of each triangle is always 1 within the unit circle.

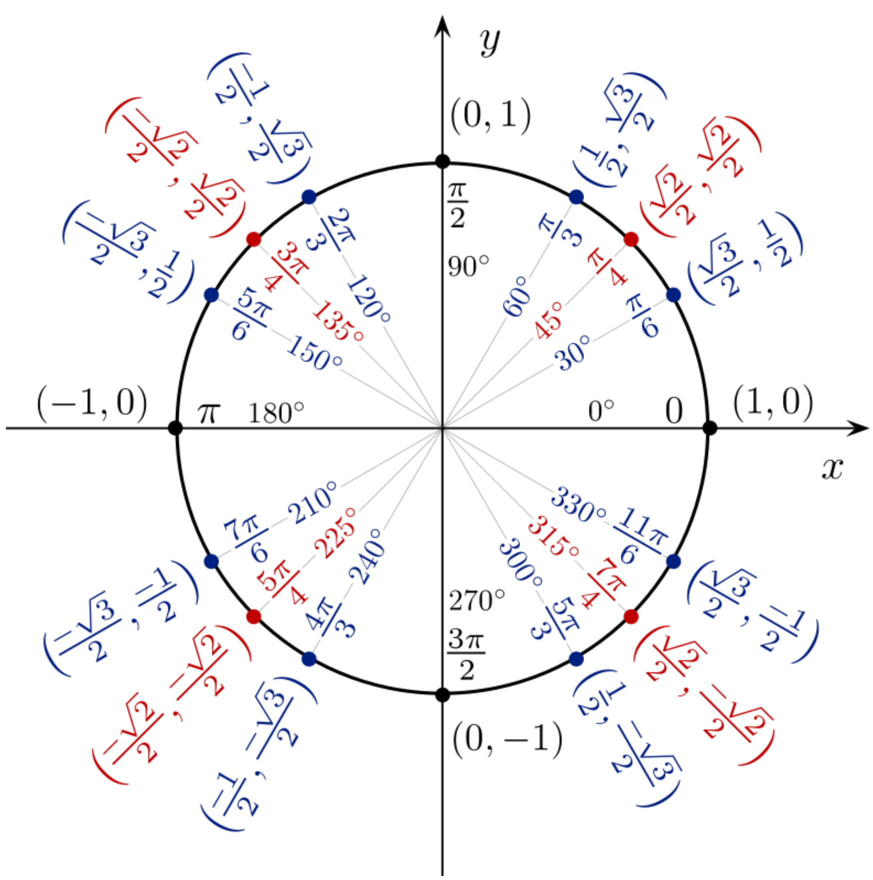
-If I want to find the $\cos(30^o)$, I remember that cosine is $\frac{opposite}{adjacent}$ and therefore, looking at the diagram, I can see that $\cos(30^o) = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$ from the black triangle.

-You may notice that this also happens to be the x-value in the ordered pair that corresponds to 30-degrees. This is not coincidence, and that is why the unit circle is such a powerful tool.

-When it comes to using the unit circle and its ordered pairs that correspond to a certain angle, the following table holds true and may be of assistance.

$\sin\theta = y$	$\cos\theta = x$	$\tan\theta = \frac{y}{x}$
$\csc\theta = \frac{1}{y}$	$\sec\theta = \frac{1}{x}$	$\cot\theta = \frac{x}{y}$

-In a similar manner, the remaining 3 Quadrants of the unit circle can be constructed, and the completed product looks like the one below courtesy of Wikipedia.



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Let's look at calculating a few values with the unit circle

Below, we will be rationalizing all our answers if there is a radical in the denominator.

This is not technically needed; however, many teachers prefer rationalized solutions therefore, we will rationalize.

a.) $\sin\left(\frac{\pi}{3}\right)$

Since sine is the y coordinate is we go the angle $\pi/3$ in the unit circle and see that the y coordinate is ${}^{\sqrt{3}}/_{2}$.

Therefore, $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

b.) $\sec(225^o) = \frac{1}{\left(-\frac{\sqrt{2}}{2}\right)} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

c.) $\tan\left(\frac{11\pi}{6}\right) = \frac{\left(-\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \left(-\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

d.) $\cos(135^o) = -\frac{\sqrt{2}}{2}$

e.) $\cot(\pi) = \frac{-1}{0} = undefined$

f.) $\csc(240^o) = \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$