Trigonometric Equations

-Trigonometric equations form an extremely wide topic in any trigonometry course.

-There are many different types of trigonometric equations and many different methods for solving them.

-In this tutorial we will start with simple equations and work our way out.

-Please have your unit circle available, whenever possible we will use the circle to find solutions.

-There will be times that we need to use the calculator.

Simple Trigonometric Equations

-With simple equations sometimes, you only need to isolate the trigonometric term and then use inverse trig to find the solutions.

-Below, we will look at a few examples.

Example 1: Solve $-2\sin x - 1 = 0$ on the interval $[0, 2\pi)$. (give your solution in radians)

SOLUTION:

We first add 1 to both sides.

$$-2\sin x = 1$$

Divide by -2.

$$\sin x = -\frac{1}{2}$$

Now we can look at the unit circle to see where sine has this value.

If we go to the unit circle, we find that there are two values on the interval $[0, 2\pi)$ that satisfy the above equation.

$$x = \frac{7\pi}{6} \qquad x = \frac{11\pi}{6}$$

Example 2: Solve $2\cos x - \sqrt{3} = 0$ on the interval $[0, 360)$. (give your solutions in degrees)

SOLUTION:

We first add $\sqrt{3}$ to both sides.

$$2\cos x = \sqrt{3}$$

We then divide by 2.

$$\cos x = \frac{\sqrt{3}}{2}$$

Now, we can look at the unit circle, to see where cosine has this value.

If we take this to the unit circle, we get the following two values.

$$x = 30^o \quad , \quad x = 330^o$$

Example 3: Solve $4\sin x - \sqrt{3} = 2\sin x$ in the interval $[0, 2\pi)$. (give your solution in radians)

SOLUTION:

Add $\sqrt{3}$ to both sides of the equation.

$$4\sin x = 2\sin x + \sqrt{3}$$

Subtract $2\sin x$ from both sides.

$$2\sin x = \sqrt{3}$$

Divide both sides by 2.

$$\sin x = \frac{\sqrt{3}}{2}$$

We can look at the unit circle to see where sine has this value.

If we go to the unit circle, we get the following two values.

$$x = \frac{\pi}{3} \quad , \quad x = \frac{2\pi}{3}$$

Example 4: Solve $3\sin x - 2 = 7\sin x - 1$ on the interval $[0^o,\ 360^o)$. (give your answer in degrees)

SOLUTION:

We start by adding 1 to both sides of the equation.

$$3\sin x = 7\sin x + 1$$

Next, we subtract $7\sin x$ from both sides.

$$-4\sin x = 1$$

Divide both sides by -4.

$$\sin x = -\frac{1}{4}$$

This value is of course not on the unit circle, therefore, we punch it into the calculator, before doing this, make sure that your mode is in degrees. You will get the following solution.

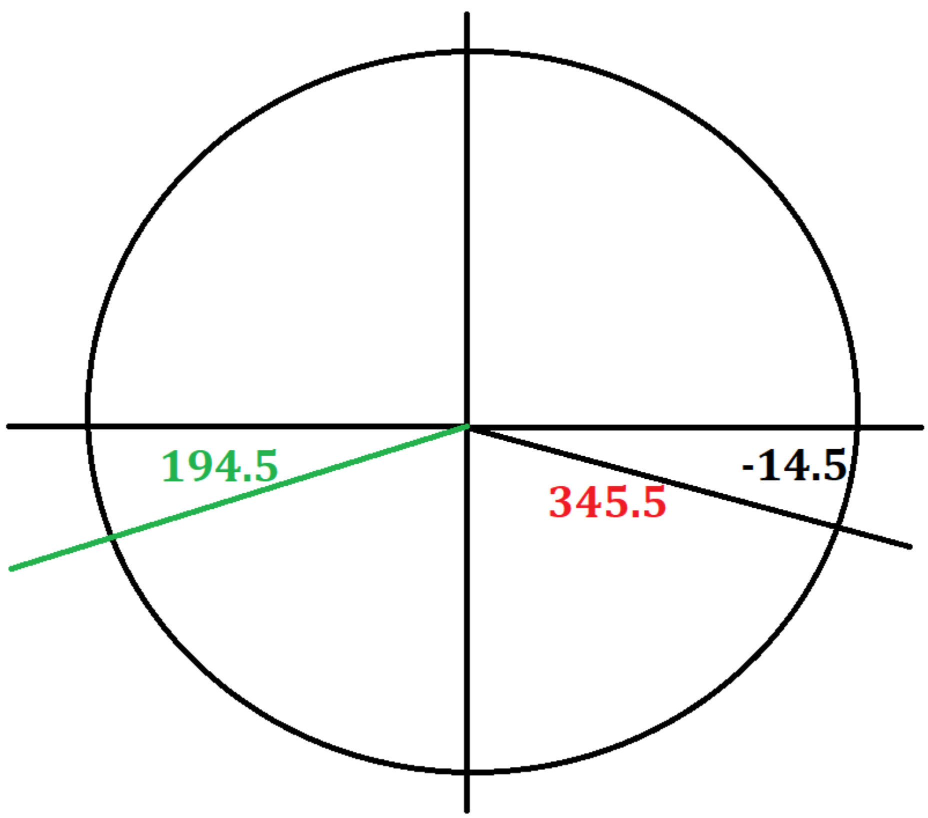
$$x = -14.47751219\ldots^{o}$$

We will round this to the nearest 10^{th} to get

$$x = -14.5^{o}.$$

This solution should look a little unusual because first its negative, and second, we only got one solution, there are technically two.

Let's look at the diagram below to see how we can get two positive solutions out of the negative one the calculator gave us.



We want to provide our answers as positive angles and we can see that 345.5^o is coterminal to -14.5^o so that means that $x = 345.5^o$ is a solution.

But, since sine is the y coordinate, we can see that there is another angle that would produce the same y value as $x = 345.5^o$.

We can find this by adding $180^o + 14.5^0 = 194.5^o$.

This means that the two solutions to this equation are as follows.

$$x = 345.5^o \quad , \quad x = 194.5^o$$

Example 5: Solve $2\sin x - 4 = -1$ on the interval $[0, 2\pi)$. (give your solution/s in radians)

SOLUTION:

Start by adding -4 to both sides.

$$2\sin x = 3$$

Divide both sides by 2.

$$\sin x = \frac{3}{2} = 1.5$$

Ordinarily, this is the step where we would look at the unit circle or consult the calculator.

However, if you punch this into your calculator, you will get an error message.

To understand what is happening, lets recall the sine function.

Remember that $\sin x$ has a range of $[1, -1]$, it continuously bounces between those two values, it never goes higher than 1 or lower than -1.

This means that $\sin x = 1.5$ cannot exist because the sine never goes past 1 to 1.5.

Therefore, there is no solution, or the solution is $\emptyset$.

Example 6: Find all degree solutions for the following equation.

$$\cos(2x - 50^o) = \frac{\sqrt{3}}{2}$$

SOLUTION: An important distinction about this problem is that ALL solutions are being asked for. Since trigonometric functions are circular, every time we wrap around 360^o we wind up at the same point. So can always get another solution by adding or subtracting 360.

If we go to the unit circle, we can see that there are two places where the cosine is $\frac{\sqrt{3}}{2}$, they are 30^o and 330^o. Therefore, we could set up the following two equations to find x.

$$2x - 50^o = 30^o \qquad 2x - 50^o = 330^o$$

But there is one problem with this, we need to have ALL solutions, so, we make the following change.

$$2x - 50^o = 30^o + 360^o k \qquad 2x - 50^o = 330^o + 360^o k$$

Where k has to be an integer.

We start by adding 50^o to both sides of each equation.

$$2x = 80^o + 360^o k \qquad 2x = 380^o + 360^o k$$

We now divide both sides of each equation by 2.

$$x = 40^o + 180^o k \qquad x = 190^o + 180^o k \qquad Where\ k\ is\ some\ integer$$

These two represent all solutions for the equation. What is cool about this is that if you plug in any integer value for k and simplify, you will have a solution to the equation.

Example 7: Find all degree solutions for the following equation.

$$\sin(3x + 30^o) = \frac{1}{2}$$

SOLUTION: An important distinction about this problem is that ALL solutions are being asked for. Since trigonometric functions are circular, every time we wrap around 360^o we wind up at the same point. So can always get another solution by adding or subtracting 360.

If we go to the unit circle, we can see that there are two places where the cosine is $\frac{1}{2}$, they are 30^o and 150^o. Therefore, we could set up the following two equations to find x.

$$3x + 30^o = 30^o \qquad 3x + 30^o = 150^o$$

But there is one problem with this, we need to have ALL solutions, so, we make the following change.

$$3x + 30^o = 30^o + 360^o k \qquad 3x + 30^o = 150^o + 360^o k$$

Where k must be an integer.

We start by subtracting 30^o to both sides of each equation.

$$3x = 360^o k \qquad 3x = 120^o + 360^o k$$

We now divide both sides of each equation by 3.

$$x = 0^o + 120^o k \qquad x = 40^o + 120^o k \qquad Where\ k\ is\ some\ integer$$

These two represent all solutions for the equation. What is cool about this is that if you plug in any integer value for k and simplify, you will have a solution to the equation.

Solving by Factoring

Example 1: Solve the following trigonometric equation on the interval $[0, 2\pi)$ in radians.

$$\sin x + 2\sin x \cos x = 0$$

SOLUTION:

We start by noticing that we can factor sin x out from the equation.

$$\sin x\,(1 + 2\cos x) = 0$$

We set each factor equal to 0 and solve.

$$\sin x = 0 \qquad 1 + 2\cos x = 0$$

$$2\cos x = -1$$

$$\cos x = -\frac{1}{2}$$

So, we now have the following two equations to solve.

$$\sin x = 0 \qquad \cos x = -\frac{1}{2}$$

Now, we use inverse trigonometry as follows.

$$x = \sin^{-1}(0) \qquad x = \cos^{-1}\left(-\frac{1}{2}\right)$$

We get the following solutions.

$$\{0, \pi\} \qquad \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

We can combine everything to state our solutions as follows.

$$\left\{0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

Example 2: Solve the following trigonometric equation on the interval $[0, 2\pi)$ in radians.

$$2\sin^2 x - \sin x - 1 = 0$$

SOLUTION:

Since this equation looks a little quadratic, we are going to use substitution so solve this equation.

Let's say that $u = \sin x$

Once we make the substitution, we get the following quadratic equation.

$$2u^2 - u - 1 = 0$$

We will solve this equation by factoring below.

$$2u^2 - u - 1 = 0$$

$$(2u + 1)(u - 1) = 0$$

$$2u + 1 = 0 \qquad u - 1 = 0$$

$$u = -\frac{1}{2} \qquad u = 1$$

Now, we back substitute because $u = \sin x$ we get the following two equations.

$$\sin x = -\frac{1}{2} \qquad \sin x = 1$$

We apply inverse trig for the following.

$$x = \sin^{-1}\left(-\frac{1}{2}\right) \qquad x = \sin^{-1}(1)$$

From the unit circle, we get the following solutions.

$$\left\{\frac{11\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{2}\right\}$$

More Complicated Trigonometric Equations

-Some trigonometric equations require multiple techniques to solve.

-The techniques most often used are trig identities and algebra techniques such as factoring and clearing fractions.

-Below, we will look at some examples of problems that are a little more interesting.

Example 1: Solve the following trigonometric equation on the interval $[0, 360^o)$.

$$\sqrt{2}\csc x + 5 = 3$$

SOLUTION:

We start this problem by isolating the trigonometric term in the following table.

Subtract 5 from both sides.	$\sqrt{2}\csc x = -2$
Divide both sides by $\sqrt{2}$.	$\csc x = -\frac{2}{\sqrt{2}}$
Apply the reciprocal identity.	$\sin x = -\frac{\sqrt{2}}{2}$

We apply inverse trigonometry $x = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ and go to the unit circle to get the following solutions.

$$x = 225^o \qquad x = 315^o$$

Example 2: Solve the following equation on the interval $[0, 2\pi)$.

$$4\sin x - 2\csc x = 0$$

SOLUTION: We will solve this problem in the table below.

We apply the reciprocal identity.	$4\sin x - 2\left(\frac{1}{\sin x}\right) = 0$
Clear all the fractions.	$\sin x\left[4\sin x - 2\left(\frac{1}{\sin x}\right) = 0\right]$
Distribute	$4\sin^2 x - 2 = 0$
Add 2 to both sides	$4\sin^2 x = 2$
Divide both sides by 4	$\sin^2 x = \frac{1}{2}$
Take the square root of both sides.	$\sqrt{\sin^2 x} = \sqrt{\frac{1}{2}}$

Simplify	$\sin x = \pm\frac{1}{\sqrt{2}}$
Rationalize.	$\sin x = \pm\frac{\sqrt{2}}{2}$

What we really have is the two separate equations below.

$$\sin x = \frac{\sqrt{2}}{2} \qquad \sin x = -\frac{\sqrt{2}}{2}$$

From the unit circle, we get the following solutions.

$$\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\} \qquad \left\{\frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$

Although we haven't it till this point, you always want to check your solutions to make sure that they work.

All these solutions check out, therefore the solutions to this equation are as follows.

$$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$

Example 3: Solve the following trigonometric equation on the interval [0°, 360°).

$$\sin 2x - \cos x = 0$$

SOLUTION: We will solve this equation in the table below.

Apply the Double-Angle Identity	$2\sin x\cos x - \cos x = 0$
Factor out the cosine	$\cos x\,(2\sin x - 1) = 0$

Now that the equation is factored, we can split it into the following two equations.

$$\cos x = 0 \qquad 2\sin x - 1 = 0$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2}$$

This leaves us the following two equations to solve.

$$\cos x = 0 \qquad \sin x = \frac{1}{2}$$

From the unit circle, we get the following solutions.

$$\{90^o, 270^o, 30^o, 150^o\}$$

Example 4: Solve the following trigonometric equation on the interval $[0, 2\pi)$.

$$\cos 2x - 3\sin x - 2 = 0$$

SOLUTION: We will solve this equation in the following table.

Apply the Double-Angle Identity	$(1 - 2\sin^2 x) - 3\sin x - 2 = 0$
Combine like terms	$-2\sin^2 x - 3\sin x - 1 = 0$
Multiply by -1 (optional)	$2\sin^2 x + 3\sin x + 1 = 0$

At this point, you may recognize this problem as one that requires substitution, so we can say that $u = \sin x$. After our substitution we have the following equation which we will factor and solve.

$$2u^2 + 3u + 1 = 0$$

$$(2u + 1)(u + 1) = 0$$

$$2u + 1 = 0 \qquad u + 1 = 0$$

$$u = -\frac{1}{2} \qquad u = -1$$

We back substitute to get the following two equations.

$$\sin x = -\frac{1}{2} \qquad \sin x = -1$$

From the unit circle, we get the following solutions.

$$\left\{\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{3\pi}{2}\right\}$$

Example 5: Solve the following trigonometric equation on the interval $[0, 2\pi)$.

$$2\cos^2 x + \sin x - 1 = 0$$

SOLUTION:

This problem looks like substitution might work but it needs to be altered a little bit before that is possible.

Let's make this change in the table below.

Apply the Pythagorean Identity	$2(1 - \sin^2 x) + \sin x - 1 = 0$
Distribute	$2 - 2\sin^2 x + \sin x - 1 = 0$

Combine like terms	$-2\sin^2 x + \sin x + 1 = 0$
Multiply by -1 (optional)	$2\sin^2 x - \sin x - 1 = 0$

Now that this is complete our new equation reads as $2\sin^2 x - \sin x - 1 = 0$ and this can be solved using u-substitution.

If we say that $u = \sin x$, then we can make the following substitution. Then we can factor and solve for u.

$$2u^2 - u - 1 = 0$$

$$(2u + 1)(u - 1) = 0$$

$$2u + 1 = 0 \qquad u - 1 = 0$$

$$u = -\frac{1}{2} \qquad u = 1$$

If we back substitute, we get the following two equations.

$$\sin x = -\frac{1}{2} \qquad \sin x = 1$$

Upon going to the unit circle, we should have the following solutions.

$$\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$$

Example 6: Solve the following trigonometric equation on the interval $[0^o, 360^o)$.

$$6\cos x + 7\tan x = \sec x$$

SOLUTION:

With a problem like this it is often helpful to start with two basic steps.

1- Turn everything into sines and cosines.
2- Clear all fractions.

While taking these steps will not necessarily solve your problem, they usually put it in a form that is easier to solve.

Turn everything into sines and cosines.	$6\cos x + 7\frac{\sin x}{\cos x} = \frac{1}{\cos x}$
Clear all fractions.	$\cos x\left[6\cos x + 7\frac{\sin x}{\cos x} = \frac{1}{\cos x}\right]$

Simplify	$6\cos^2 x + 7\sin x = 1$

Now we have changed our problem into something recognizable, we will take steps to solve in below.

Apply the Pythagorean Identity	$6(1 - \sin^2 x) + 7\sin x = 1$
Distribute	$6 - 6\sin^2 x + 7\sin x = 1$
Subtract 1 from both sides	$5 - 6\sin^2 x + 7\sin x = 0$
Reorder	$-6\sin^2 x + 7\sin x + 5 = 0$
Multiply by -1 (optional)	$6\sin^2 x - 7\sin x - 5 = 0$

Now that our equation is in quadratic form, we can make the substitution $u = \sin x$ and solve the resulting quadratic equation by factoring.

$$6u^2 - 7u - 5 = 0$$

$$(3u - 5)(2u + 1) = 0$$

$$3u - 5 = 0 \qquad 2u + 1 = 0$$

$$u = \frac{5}{3} \qquad u = -\frac{1}{2}$$

By back substituting, we get the following two equations.

$$\sin x = \frac{5}{3} \qquad \sin x = -\frac{1}{2}$$

We can conclude that $\sin x = \frac{5}{3}$ has no solutions because 5/3 is greater than one. We can get the solution to the other equation using the unit circle.

From the unit circle, we get the following solutions.

$$(210^o, 330^o)$$

Example 7: Solve the following trigonometric equation in the interval $[0^o, 360^o)$.

$$23\csc^2 x - 22\cot x \csc x - 15 = 0$$

SOLUTION:

We can reduce this equation into something that can be solved in the following table.

Turn everything into sines and cosines.	$23\frac{1}{\sin^2 x} - 22\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right) - 15 = 0$

Simplify	$\frac{23}{\sin^2 x} - 22\left(\frac{\cos x}{\sin^2 x}\right) - 15 = 0$
Clear Fractions	$\sin^2 x\left[\frac{23}{\sin^2 x} - 22\left(\frac{\cos x}{\sin^2 x}\right) - 15 = 0\right]$
Distribute	$23 - 22\cos x - 15\sin^2 x = 0$
Apply the Pythagorean Identity.	$23 - 22\cos x - 15(1 - \cos^2 x) = 0$
Distribute	$23 - 22\cos x - 15 + 15\cos^2 x = 0$
Combine like terms	$8 - 22\cos x + 15\cos^2 x = 0$
Reorganize	$15\cos^2 x - 22\cos x + 8 = 0$

Now that our equation is in a quadratic form, we can the make the substitution $u = \cos x$. We can obtain and solve the quadratic equation below.

$$15u^2 - 22u + 8 = 0$$

$$(3u - 2)(5u - 4) = 0$$

$$3u - 2 = 0 \qquad 5u - 4 = 0$$

$$u = \frac{2}{3} \qquad u = \frac{4}{5}$$

We can back substitute to get the following two equations.

$$\cos x = \frac{2}{3} \qquad \cos x = \frac{4}{5}$$

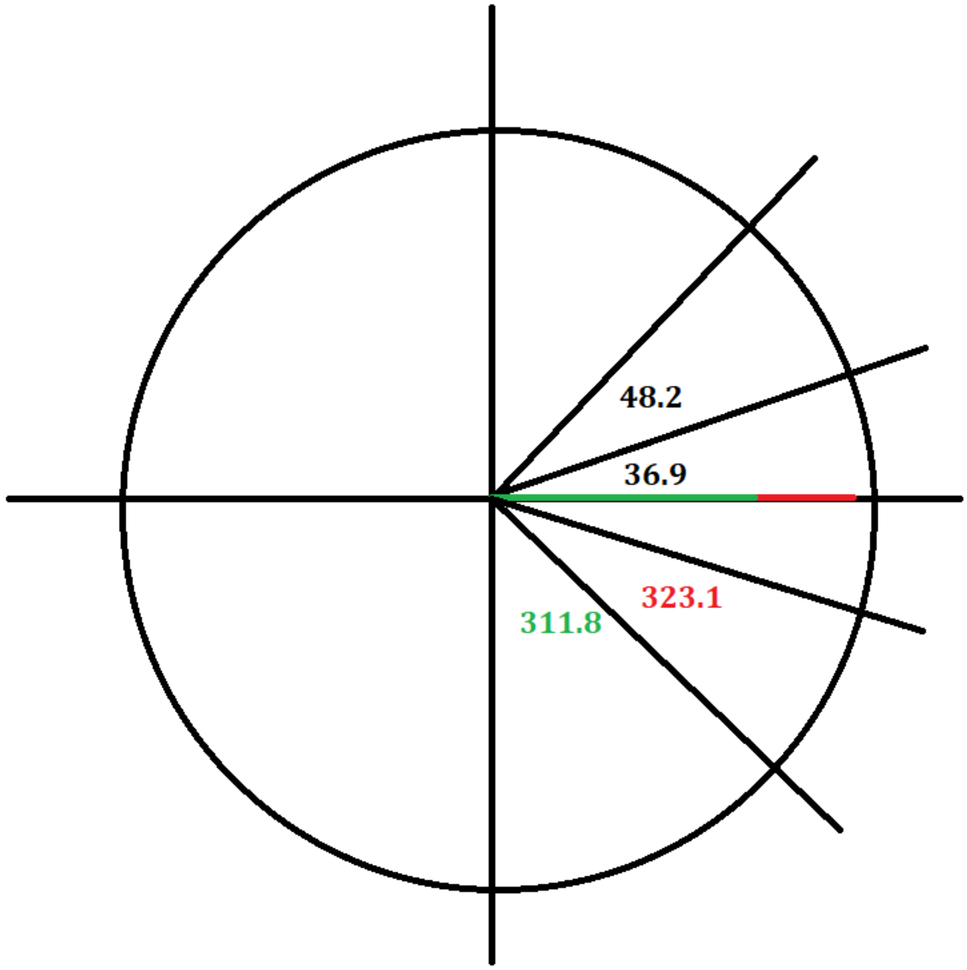
We can apply arcsine to get the following two equations.

$$x = \cos^{-1}\left(\frac{2}{3}\right) \qquad x = \cos^{-1}\left(\frac{4}{5}\right)$$

Since we cannot use the unit circle, we will use our calculators. (please make sure that you are in degree mode)

$$x = 48.2^o \qquad x = 36.9^o$$

We remember that the calculator only gives us one solution so technically we have to find the remaining two solutions for this problem. We remember that cosine is related to the x (horizontal) value. In the following diagram it is demonstrated how we found the remaining solutions.



Therefore, the solutions are as follows.

$$\{36.9^o, 48.2^o, 311.8^o, 323.1^o\}$$